Electromagnetics, SEGE4, Lessons

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MASTER 2R SEGE International
Systèmes Electroniques et Génie Electrique

12 septembre 2016

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1 Reflection from a dielectric medium

1.1 Maxwell's equations

1.1.1 The 4 Maxwell's equations

The laws of electricity and magnetism were established in 1876 by James Clerk Maxwell (1831-1879). In three-dimensional vector notation, the Maxwell equations are

$$\overrightarrow{\text{rot}}\overrightarrow{H} = \frac{\partial \overrightarrow{D}}{\partial t} + \overrightarrow{J} \tag{1.1}$$

$$\overrightarrow{\text{rot}}\overrightarrow{E} = -\frac{\partial \overrightarrow{B}}{\partial t} \tag{1.2}$$

$$\text{div}\overrightarrow{B} = 0 \tag{1.3}$$

$$\text{div}\overrightarrow{D} = \rho \tag{1.4}$$

It is important to note that the four Maxwell equations depend both on the time t and the position vector $\overrightarrow{r} = x\,\overrightarrow{x} + y\,\overrightarrow{y} + z\,\overrightarrow{z}$.

Eq. (1.1) is Ampere's law or the generalized Ampère circuit law. Eq. (1.2) is Faraday's law or Faraday's magnetic induction law. Eq. (1.3) is Coulomb's law or Gauss' law for electric fields. Eq. (1.4) is Coulomb's law or Gauss' law for magnetic fields. Maxwell's contribution to the laws of electricity and magnetism is the addition of the displacement term $\partial\overrightarrow{D}/\partial t$ in Ampère's law (1.1).

For more clarity, the notations are reported in table 1.1.1. The couple $(\overrightarrow{E}, \overrightarrow{H})$ are named the electromagnetic field.

In Cartesian coordinates $(\overrightarrow{x}, \overrightarrow{y}, \overrightarrow{z})$, the operator nabla $\overrightarrow{\nabla}$ is defined as

$$\overrightarrow{\nabla} = \frac{\partial}{\partial x}\overrightarrow{x} + \frac{\partial}{\partial y}\overrightarrow{y} + \frac{\partial}{\partial x}\overrightarrow{z} \tag{1.5}$$

Then, in Cartesian coordinates, the scalar operator $\text{div}\overrightarrow{A} = \overrightarrow{\nabla}\cdot\overrightarrow{A}$ (dot product), where $\overrightarrow{A} = A_x\overrightarrow{x} + A_y\overrightarrow{y} + A_z\overrightarrow{z} = (A_x, A_y, A_z)$, is expressed as

$$\text{div}\overrightarrow{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} \tag{1.6}$$

Variable	Name	Unity
$\overrightarrow{E}$	Electric field	V/m
$\overrightarrow{H}$	Magnetic field	A/m
$\overrightarrow{D}$	Electric displacement	C/m^2
$\overrightarrow{B}$	Magnetic flux density	Wb/m^2
$\overrightarrow{J}$	Electric current density	A/m^2
ρ	Electric charge density	C/m^3

Table 1.1 – Variables involved in the Maxwell equations.

Moreover, the vectorial operator $\overrightarrow{\mathrm{rot}}\overrightarrow{A} = \overrightarrow{\nabla} \wedge \overrightarrow{A}$ (cross product) is expressed as

$$\overrightarrow{\mathrm{rot}}\overrightarrow{A} = \begin{vmatrix} \overrightarrow{x} & \overrightarrow{y} & \overrightarrow{z} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix} = \overrightarrow{x}\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) + \overrightarrow{y}\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right) + \overrightarrow{z}\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right) \quad (1.7)$$

From the Maxwell equations, Eqs. (1.3) and (1.4) are scalar, whereas Eqs. (1.1) and (1.2) are vectorial, thus 8 scalar equations. In fact, these 8 equations are not independent. Indeed, taking the div of Eq. (1.1) and since $\mathrm{div}(\overrightarrow{\mathrm{rot}}\overrightarrow{A}) = 0$ for any vector $\overrightarrow{A}$, then

$$\mathrm{div}\overrightarrow{J} = -\frac{\partial \rho}{\partial t} \quad (1.8)$$

1.1.2 Constitutive relations in free space

The Maxwell's equations are fundamental laws governing the behavior of electromagnetic fields in free space and in media. Free space [1] is characterized by the constitutive relations :

$$\overrightarrow{D} = \epsilon_0 \overrightarrow{E} \quad (1.9a)$$

$$\overrightarrow{B} = \mu_0 \overrightarrow{H} \quad (1.9b)$$

where

$$\begin{cases} \epsilon_0 = 1/(36\pi \times 10^9) \approx 8.85 \times 10^{-12} \text{ F/m} \\ \mu_0 = 4\pi \times 10^{-7} \text{ H/m} \end{cases} \quad (1.10)$$

are, respectively, the permittivity and the permeability in free space. Giving the velocity of light in free space being $c = 3 \times 10^8$ m/s, the permittivity $\epsilon_0 = 1/(\mu_0 c^2)$, which follows from the dispersion relation as derived below.

1.1.3 Wave equation

The Maxwell equations in differential form are valid at all times for every point in space. First we shall investigate solutions to the Maxwell equations in regions devoid of source, namely

1. Free space is a medium assumed to be **linear**, **homogeneous** and **isotropic**.

in regions where $\overrightarrow{J} = \overrightarrow{0}$ and $\rho = 0$. This of course does not mean that there is no source anywhere in all space. Sources must exist outside the regions of interest in order to produce fields in these regions. Thus in source-free regions in free space, The Maxwell equations become

$$\overrightarrow{\nabla} \wedge \overrightarrow{H} = \epsilon_0 \frac{\partial \overrightarrow{E}}{\partial t} \tag{1.11}$$

$$\overrightarrow{\nabla} \wedge \overrightarrow{E} = -\mu_0 \frac{\partial \overrightarrow{H}}{\partial t} \tag{1.12}$$

$$\overrightarrow{\nabla} \cdot \overrightarrow{H} = 0 \tag{1.13}$$

$$\overrightarrow{\nabla} \cdot \overrightarrow{E} = 0 \tag{1.14}$$

In the form of scalar partial differential equations, we have from Eqs. (1.6) and (1.7)

$$\begin{cases} \frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z} = \epsilon_0 \frac{\partial E_x}{\partial t} & \text{(a)} \\ \frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x} = \epsilon_0 \frac{\partial E_y}{\partial t} & \text{(b)} \\ \frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} = \epsilon_0 \frac{\partial E_z}{\partial t} & \text{(c)} \end{cases} \tag{1.15}$$

$$\begin{cases} \frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} = -\mu_0 \frac{\partial H_x}{\partial t} & \text{(a)} \\ \frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x} = -\mu_0 \frac{\partial H_y}{\partial t} & \text{(b)} \\ \frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} = -\mu_0 \frac{\partial H_z}{\partial t} & \text{(c)} \end{cases} \tag{1.16}$$

$$\frac{\partial H_x}{\partial x} + \frac{\partial H_y}{\partial y} + \frac{\partial H_z}{\partial z} = 0 \tag{1.17}$$

$$\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} = 0 \tag{1.18}$$

A wave equation for $\overrightarrow{E}$ can be derived by eliminating $\overrightarrow{H}$ from Eqs. (1.15) and (1.16). Taking time derivatives of Eq. (1.15a) and substituting Eqs. (1.16c) and (1.16b), we have

$$\begin{aligned} \mu_0 \epsilon_0 \frac{\partial^2 E_x}{\partial t^2} &= -\frac{\partial}{\partial y}\left(\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y}\right) + \frac{\partial}{\partial z}\left(\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x}\right) \\ &= \frac{\partial^2 E_x}{\partial y^2} + \frac{\partial^2 E_x}{\partial z^2} - \frac{\partial^2 E_y}{\partial y \partial x} - \frac{\partial^2 E_z}{\partial z \partial x} \\ &= \frac{\partial^2 E_x}{\partial y^2} + \frac{\partial^2 E_x}{\partial z^2} + \frac{\partial^2 E_x}{\partial x^2} \text{ from } \frac{[\partial \text{Eq. (1.18)}]}{\partial x} \end{aligned} \tag{1.19}$$

Thus, we obtain the following equations for the three components of $\overrightarrow{E}$:

$$\begin{cases} \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} - \mu_0 \epsilon_0 \frac{\partial^2}{\partial t^2}\right) E_x = 0 \\ \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} - \mu_0 \epsilon_0 \frac{\partial^2}{\partial t^2}\right) E_y = 0 \\ \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} - \mu_0 \epsilon_0 \frac{\partial^2}{\partial t^2}\right) E_z = 0 \end{cases} \tag{1.20}$$

Introducing the scalar Laplacian operator $\nabla^2 = \overrightarrow{\nabla} \cdot \overrightarrow{\nabla}$ in Cartesian coordinates

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \tag{1.21}$$

we have

$$\nabla^2 \overrightarrow{E} - \mu_0 \epsilon_0 \frac{\partial^2 \overrightarrow{E}}{\partial t^2} = \overrightarrow{0} \tag{1.22}$$

This is known as the Helmholtz wave equation.

1.1.4 Wave solution

Solutions of the wave (1.22) that satisfy all Maxwell equations are electromagnetic waves. We shall now study a solution to Eq. (1.19) assuming $E_x = E_y = 0$. Let E_x be a function only of z and t and independent of x and y. The electric field vector can be written as

$$\overrightarrow{E} = E_x(z,t)\overrightarrow{x} \tag{1.23}$$

The wave equation it satisfied follows from Eq. (1.22), which becomes

$$\frac{\partial^2 E_x}{\partial z^2} - \mu_0 \epsilon_0 \frac{\partial^2 E_x}{\partial t^2} = 0 \tag{1.24}$$

The simplest solution to Eq. (1.24) takes the form

$$\overrightarrow{E} = E_x(z,t)\overrightarrow{x} = E_0 \cos\left(kz - \omega t\right)\overrightarrow{x} \tag{1.25}$$

Substituting Eq. (1.25) into Eq. (1.24), we find that the following equation, called the dispersion relation, must be satisfied :

$$k^2 = \omega^2 \mu_0 \epsilon_0 \tag{1.26}$$

The dispersion relation provides an important connection between the spatial frequency k and the temporal frequency ω.

There are two points of view useful in the study of a space-time varying quantity such as $E_x(z,t)$. The temporal view point is to examine the time variations at fixed points in space. The spatial view point is to examine spatial variations at fixed times, a process that amounts to taking a series of pictures.

1.1.5 Time representation

From the temporal view point, we first fix our attention on particular point in space, say $z = 0$. We then have the electric field $E_x(z = 0, t) = E_0 \cos(\omega t)$. Plotted as a function of time in Fig. 1.1, we find that the waveform repeats itself in time as $\omega t = 2m\pi$ for any integer m. The period is defined as the time T, for which $\omega T = 2\pi$. The number of periods in a time of one second is the frequency f defined as $f = 1/T$, which gives

$$f = \frac{\omega}{2\pi} \tag{1.27}$$

The unity for the frequency f is Hertz (Hz) with $1\text{Hz} = 1\ \text{s}^{-1}$, which is equal to the number of cycles per second. Since, $\omega = 2\pi f$, ω is the angular frequency of the wave.

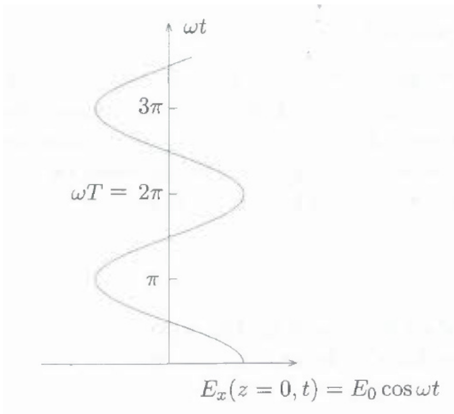

FIGURE 1.1 – Electric field strength as a function of ωt at $z = 0$.

The temporal frequency ω characterizes the wave in time. We plot in Fig. 1.2a $E_x(z = 0, t)$ as a function of t instead of ωt. Let there be one period within the time interval of 1 second. Thus, $f = f_0 = 1$ Hz, and we let $\omega = \omega_0 = 2\pi$ rad/s. In Fig. 1.2b, we plot $\omega = 2\omega_0$; there are two periods in a time interval of one second and the period in time is 0.5 second. In Fig. 1.2c, $\omega = 3\omega_0$ and there are three periods in one second.

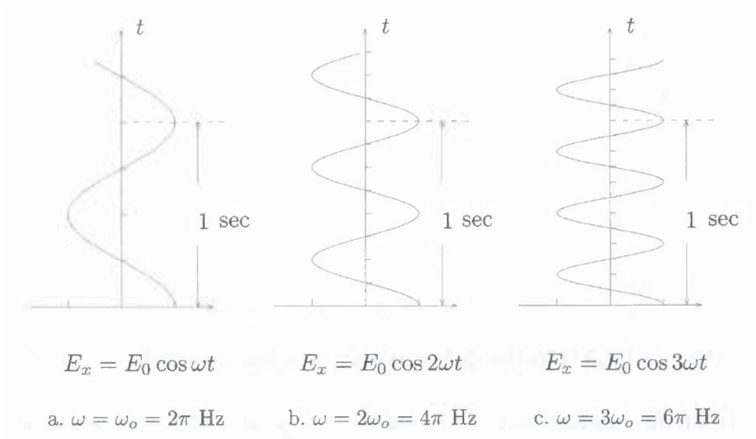

FIGURE 1.2 – Electric field strength as a function of t for different angular frequencies ω.

1.1.6 Space representation

To examine behavior from spatial view point, we let $\omega t = 0$ and plot $E_x(z, t = 0)$ in Fig. 1.3. The waveform repeats itself in space when $kz = 2m\pi$ for integer values of m. The spatial frequency k characterizes the variation of the wave in space. The wavelength is defined as the distance for which $k\lambda = 2\pi$. Thus, $\lambda = 2\pi/k$, or

$$k = \frac{2\pi}{\lambda} \tag{1.28}$$

We call k the spatial frequency of the wavenumber which is equal to the number of wavelengths in a distance of 2π and has the dimension of inverse length.

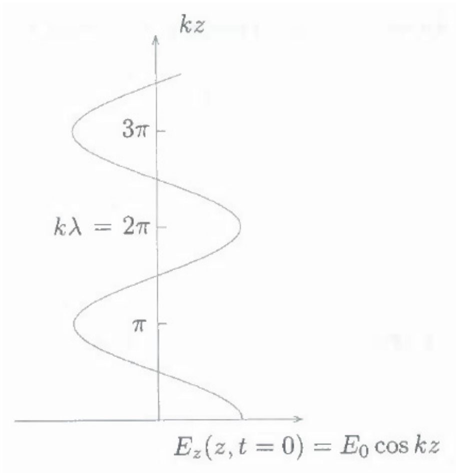

FIGURE 1.3 – Electric field strength as a function of kz for $t = 0$.

To further understand the meaning of k as a spatial frequency, we plot in Fig. 1.4a $E_x(z, t = 0)$ as a function of z instead of kz. Let there be one period within the wavelength of 1 meter. We defined $K_0 = 2\pi$ rad/m. Thus $k = 1K_0 = 2\pi$ rad/m. In Fig 1.4b, we plot $k = 2K_0$; there are two periods in a spatial distance of one meter and the wavelength is $2\pi/k = 2\pi/(2K_0) = 0.5$ meter. In Fig 1.4c, $k = 3K_0$; there are three periods in one meter.

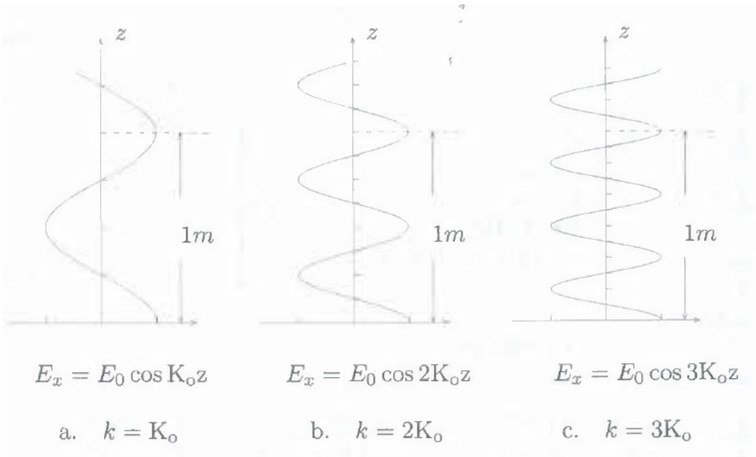

FIGURE 1.4 – Electric field strength as a function of z for with different spatial frequency.

Similar to the unit in Hz which is cycles per second in temporal variation, K_0 is cycles per meter in spatial variation. For a wave that has a spatial frequency of one period in one meter distance, $k = 1K_0$. An electromagnetic wave in free space with $k = 5K_0$ has five spatial periods in a distance of one meter. From the dispersion relation for electromagnetic waves, the spatial frequency k and and the temporal angular frequency ω are related by the velocity of light as $k = \omega/c$. In free space, the conversion factor is $c = 1/\sqrt{\mu_0\epsilon_0} = 3 \times 10^8$ m/s. Thus, for a spatial frequency of $1K_0$, the corresponding temporal frequency is $f = cK_0/(2\pi) = c = 300$ MHz.

1.1.7 Phase velocity

In Fig. 1.5, we plot $E_x(z,t)$ at two progressive times $\omega t = \pi/2$ and $\omega t = \pi$. We observe that the electric field vector at A appears to be propagating along the $\overrightarrow{z}$ direction as time progresses. The velocity of propagation v_p is determined from $kz - \omega t =$ constant, which gives

$$v_p = \frac{dz}{dt} = \frac{\omega}{k} \tag{1.29}$$

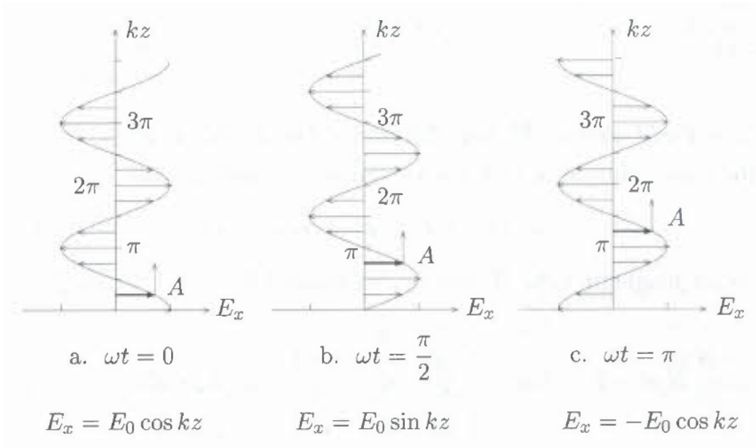

FIGURE 1.5 – Electric field strength as a function of kz at different times.

We call v_p the phase velocity. By virtue of the dispersion relation (1.26), we see that $v_p = 1/\sqrt{\mu_0\epsilon_0}$, which is equal to the velocity of light in free space.

The spatial frequency k, is according to the dispersion relation, directly related to the temporal frequency ω by the phase delay

$$\phi_p = \frac{k}{\omega} = \sqrt{\mu_0\epsilon_0} \tag{1.30}$$

which determines how much time it takes for the wave to propagate on a unit distance. In free space, $\phi_p = 10^{-8}/3$ s/m or it takes 3.3 ns for an electromagnetic wave to travel the distance of one meter.

1.1.8 Electric and magnetic field vectors

For the wave solution in Eq. (1.25) for electric field vector

$$\overrightarrow{E} = E_x(z,t)\overrightarrow{x} = E_0 \cos(kz - \omega t)\overrightarrow{x} \tag{1.31}$$

the vector magnetic field $\overrightarrow{H}$ can be determined from Eq. (1.12). We find

$$\mu_0 \frac{\partial \overrightarrow{H}}{\partial t} = -\overrightarrow{\nabla} \wedge \overrightarrow{E} = - \begin{vmatrix} \overrightarrow{x} & \overrightarrow{y} & \overrightarrow{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ E_x & 0 & 0 \end{vmatrix} = -\overrightarrow{y}\frac{\partial E_x}{\partial z} - \overrightarrow{z} \underbrace{\frac{\partial E_x}{\partial y}}_{=0 \text{ why ?}} = -\overrightarrow{y}\frac{\partial E_x}{\partial z}$$
$$= E_0 k \sin(kz - \omega t)\, \overrightarrow{y} \tag{1.32}$$

The magnetic field vector $\overrightarrow{H}$ is then

$$\overrightarrow{H} = \frac{k\overrightarrow{y}}{\mu_0} E_0 \int \sin(kz - \omega t)\, dt = \frac{k}{\omega \mu_0} E_0 \cos(kz - \omega t)\, \overrightarrow{y} \tag{1.33}$$

Eqs. (1.31) and (1.32) satisfy all the Maxwell equations (1.11), (1.12), (1.13) and (1.14).

Write the amplitude of the magnetic field vector $\overrightarrow{H}$ as H_0

$$\overrightarrow{H} = H_y(z,t)\overrightarrow{y} = H_0 \cos(kz - \omega t)\, \overrightarrow{y} \tag{1.34}$$

where $H_0 = E_0/\eta$ and $\eta = \sqrt{\mu_0/\epsilon_0} = 120\pi$ is called the free-space impedance. The electromagnetic wave is propagating in the positive $\overrightarrow{z}$ direction because as time t increases, z must increase in order to maintain a constant phase $kz - \omega t$. The field vectors of the electromagnetic wave are transversal to the direction of propagation an lie in the $xy-$plane, on which the phase $kz - \omega t$ of the wave is a constant. Since the phase front of the wave is the xy-plane, we call the electromagnetic wave as represented by Eqs. (1.31) and (1.34) a plane wave. See Fig. 1.6.

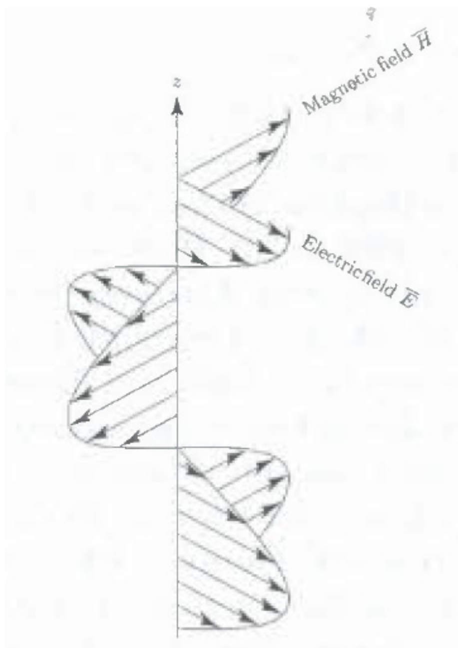

FIGURE 1.6 – Electric and magnetic field vectors of an electromagnetic wave.

1.2 Polarization

1.2.1 Introduction

The polarization of a wave is conventionally defined by the time variation of the tip of the electric field $\overrightarrow{E}$ at a fixed point in space. For example :

- If the tip moves along a straight line, the wave is then linearly polarized.
- If the tip moves along a circle, the wave is then circularly polarized.
- If the tip moves along an ellipse, the wave is then elliptically polarized.

Considering the following wave solution

$$\begin{aligned} \overrightarrow{E} &= E_x \overrightarrow{x} + E_y \overrightarrow{y} \\ &= \cos(kz - \omega t)\overrightarrow{x} + E_{0y}\cos(kz - \omega t + \delta)\overrightarrow{y} \end{aligned} \tag{1.35}$$

Note that $E_z = 0$ because the wave propagates in the $+\overrightarrow{z}$ direction.

From the temporal point of view ($z = 0$), we have

$$\overrightarrow{E} = \underbrace{\cos(\omega t)}_{E_x}\overrightarrow{x} + \underbrace{E_{0y}\cos(\omega t - \delta)}_{E_y}\overrightarrow{y} \tag{1.36}$$

We now study the polarization of the following special cases :

1. $\delta = 2n\pi$, where n is an integer, we have
$$\overrightarrow{E} = \cos(\omega t)\overrightarrow{x} + E_{0y}\cos(\omega t)\overrightarrow{y} \Rightarrow E_y = E_{0y}E_x \tag{1.37}$$
The tip of the electric field vector moves along a line as shown in figure 1.7(a). The wave is **linearly** polarized.
2. $\delta = (2n+1)\pi$, we have
$$\overrightarrow{E} = \cos(\omega t)\overrightarrow{x} - E_{0y}\cos(\omega t)\overrightarrow{x} \Rightarrow E_y = -E_{0y}E_x \tag{1.38}$$
The tip of the electric field vector moves along a line as shown in figure 1.7(b). The wave is **linearly** polarized.
3. $\delta = \pi/2$ and $E_{0y} = 1$, we have
$$\overrightarrow{E} = \cos(\omega t)\overrightarrow{x} + \sin(\omega t)\overrightarrow{y} \Rightarrow E_x^2 + E_y^2 = 1 \tag{1.39}$$
In addition, as t increases, E_x decreases whereas E_y increases. As shown in figure 1.7(c), the wave is **right-hand circularly** polarized.
4. $\delta = -\pi/2$ and $E_{0y} = 1$, we have
$$\overrightarrow{E} = \cos(\omega t)\overrightarrow{x} - \sin(\omega t)\overrightarrow{y} \Rightarrow E_x^2 + E_y^2 = 1 \tag{1.40}$$
In addition, as t increases, E_x decreases whereas E_y decreases. As shown in figure 1.7(d), the wave is **left-hand circularly** polarized.
5. $\delta = \pm\pi/2$ and $E_{0y} \neq 1$, we have
$$\overrightarrow{E} = \overrightarrow{x}\cos(\omega t)\overrightarrow{x} \pm E_{0y}\sin(\omega t)\overrightarrow{y} \Rightarrow E_x^2 + \frac{E_y^2}{E_{0y}^2} = 1 \tag{1.41}$$
As shown in figure 1.7(e)-(f), The wave is **right-hand elliptically** polarized for $\delta = \pi/2$ and **left-hand elliptically polarized** for $\delta = -\pi/2$, respectively.

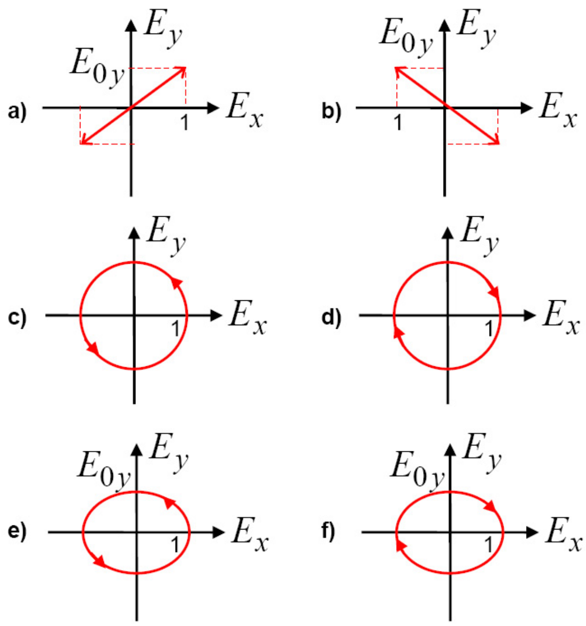

FIGURE 1.7 – Different states of polarization.

1.2.2 More general cases

1.2.2.1 Elliptical polarization with a rotation

In general, a polarized wave has en elliptical polarization. The electric field is then

$$\overrightarrow{E} = E_{0x}\cos(\omega t - \delta_0)\overrightarrow{x} \pm E_{0y}\sin(\omega t - \delta_0)\overrightarrow{y} \Rightarrow \frac{E_x^2}{E_{0x}^2} + \frac{E_y^2}{E_{0y}^2} = 1 \tag{1.42}$$

As shown at the top of figure 1.8, we see that E_{0x} is the major axis of the ellipse and E_{0y} the minor axis. With the plus sign, the wave is right-hand elliptically polarized, whereas with the minus sign, the wave is left-hand elliptically polarized. The shape of the ellipse can be specified by an ellipticity angle χ defined as

$$\tan\chi = \pm\frac{E_{0y}}{E_{0x}} = \pm\frac{b}{a} \tag{1.43}$$

In addition, as shown at the bottom of figure 1.8, the ellipse can be undergone a rotation of an angle α. In this case, the ellipticity angle is

$$\tan\chi = \pm\frac{b'}{a'} \tag{1.44}$$

This case corresponds, with $\tau = \omega t$, to

$$\begin{cases} E_x = E_{0x}\cos(\tau + \delta_x) \\ E_y = E_{0y}\cos(\tau + \delta_y) \end{cases} \tag{1.45}$$

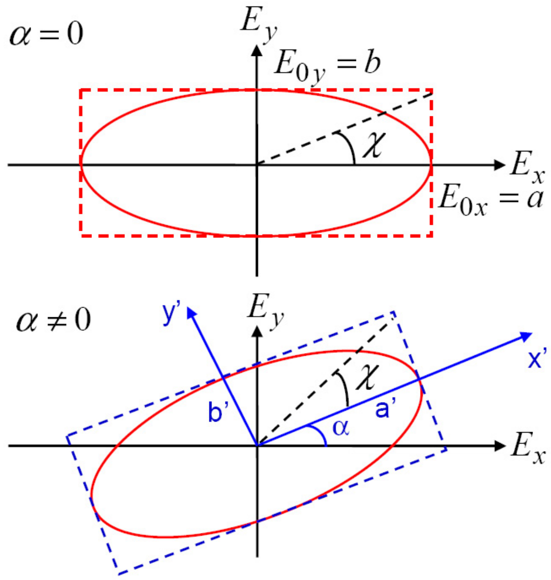

FIGURE 1.8 – General case of an elliptical polarization.

We then show (see exercise 1.5.2.1) that

$$\frac{E_x^2}{E_{0x}^2} + \frac{E_y^2}{E_{0y}^2} - 2\frac{E_x}{E_{0x}}\frac{E_y}{E_{0y}}\cos\delta = \sin^2\delta \tag{1.46}$$

with $\delta = \delta_y - \delta_x$. Comparing Eq. (1.46) with Eq. (1.42), an additional term is added related to the angle of rotation α, as shown in the next subsection.

1.2.2.2 Relations between the angles

We can show (see exercise 1.5.2.2) that the angle of rotation α and of ellipticity χ (Eq. (1.44)) are related to $a = E_{0x}$, $b = E_{0y}$ and $\delta = \delta_y - \delta_x$ by

$$\left\{\begin{aligned} \tan(2\alpha) &= \frac{2ab\cos\delta}{a^2 - b^2} \\ \sin(2\chi) &= \frac{2ab\sin\delta}{a^2 + b^2} \end{aligned}\right. \quad \text{and} \quad \left\{\begin{aligned} &\alpha \in [0;\pi[\\ &\chi \in [-\pi/4;\pi/4] \end{aligned}\right. \tag{1.47}$$

Eq. (1.47) shows that the polarization of a wave can be defined either from $a = E_{0x}$, $b = E_{0y}$ and $\delta = \delta_y - \delta_x$ or from the angles α and χ.

For example, for a linearly polarized wave, $a = E_{0x} = \pm E_{0y}$ and $\delta = 0$. Thus $\chi = 0$ and $\tan(2\alpha) \to 2(b/a)/[1 - (b/a)^2] = 2\tan\alpha/(1 - \tan^2\alpha)$. Thus $\tan\alpha = b/a = \pm 1$. Then $\alpha = \pm\pi/4$.

For example, for a circularly polarized wave, $a = E_{0x} = E_{0y}$ and $\delta = \pm\pi/2$. Then $\tan(2\alpha) = 0$, implying that $\alpha = 0$ or $\alpha = \pi/2$. In addition, $\sin(2\chi) = \pm 1$, implying that $\chi = \pm\pi/4$. By convention, the **right**-hand circularly polarization is obtained for α **positive**.

1.3 Wave propagation in a conductor medium

A conductor medium, like copper, sea, and so on, can be characterized by a LHI medium (like in free space) of permeability $\mu = \mu_0$ (no magnetic medium), permittivity $\epsilon = \epsilon_0 \epsilon_r$ with ϵ_r a real number larger than one, without charge $\rho = 0$ but $\overrightarrow{J} = \sigma \overrightarrow{E} \neq \overrightarrow{0}$. σ is the conductiity in S/m and ϵ_r the relative permittivity (dimensionless). In free space (or vacuum), $\sigma = 0$ and $\epsilon_r = 1$.

From Eqs. (1.1), (1.2), (1.3) and (1.4), the Maxwell equations become

$$\overrightarrow{\nabla} \wedge \overrightarrow{H} = \epsilon_0 \frac{\partial \overrightarrow{E}}{\partial t} + \sigma \overrightarrow{E} \tag{1.48}$$

$$\overrightarrow{\nabla} \wedge \overrightarrow{E} = -\mu_0 \frac{\partial \overrightarrow{H}}{\partial t} \tag{1.49}$$

$$\overrightarrow{\nabla} \cdot \overrightarrow{H} = 0 \tag{1.50}$$

$$\overrightarrow{\nabla} \cdot \overrightarrow{E} = 0 \tag{1.51}$$

We can show that the wave propagation is

$$\overrightarrow{\nabla}^2 \overrightarrow{E} - \mu_0 \sigma \frac{\partial \overrightarrow{E}}{\partial t} - \epsilon \mu_0 \frac{\partial^2 \overrightarrow{E}}{\partial t^2} = \overrightarrow{0} \tag{1.52}$$

It is a generalization of the wave Helmholtz equation obtained in free space (1.22). As in free space, a simple solution, but realistic, of this equation is $\overrightarrow{E}(\overrightarrow{r}, t) = \overrightarrow{E}_0 e^{-j(\omega t - \overrightarrow{k} \cdot \overrightarrow{r})}$, where $\overrightarrow{E}_0$ is a constant vector, which gives the wave polarization and $\overrightarrow{k}$ denotes the wave vector of norm the wavenumber $k = \left\| \overrightarrow{k} \right\|$ (or spatial frequency). The term $\omega t - \overrightarrow{k} \cdot \overrightarrow{r}$ is the phase and $\overrightarrow{k} \cdot \overrightarrow{r} = k_x x + k_y y + k_z z$, where $\overrightarrow{k} = (k_x, k_y, k_z)$ the components of the vector $\overrightarrow{k}$ and, $\overrightarrow{r} = (x, y, z)$ the components of the vector $\overrightarrow{r}$, which stands for the Cartesian coordinates of a point in space.

Since $\overrightarrow{E} = \overrightarrow{E}_0 e^{-j(\omega t - k_x x - k_y y - k_z z)}$, we have

$$\begin{cases} \frac{\partial \overrightarrow{E}}{\partial x} = j k_x \overrightarrow{E} \\ \frac{\partial \overrightarrow{E}}{\partial y} = j k_y \overrightarrow{E} \\ \frac{\partial \overrightarrow{E}}{\partial z} = j k_z \overrightarrow{E} \end{cases} \Rightarrow \overrightarrow{\nabla} \overrightarrow{E} = jk \overrightarrow{E} \Rightarrow \overrightarrow{\nabla} \to jk \tag{1.53}$$

and then, the operator $\overrightarrow{\nabla} = \partial/\partial x \overrightarrow{x} + \partial/\partial y \overrightarrow{y} + \partial/\partial z \overrightarrow{z}$ is then equivalent to $+j\overrightarrow{k}$. In other words, $\overrightarrow{E} = j\overrightarrow{k} \wedge \overrightarrow{E}$ and div$\overrightarrow{E} = j\overrightarrow{k} \cdot \overrightarrow{E}$, and the same equations are satisfied for $\overrightarrow{H}$. In addition,

$$\frac{\partial \overrightarrow{E}}{\partial t} = -j\omega \overrightarrow{E} \Rightarrow \frac{\partial}{\partial t} \to -j\omega \tag{1.54}$$

Thus, From Eqs. (1.51) and (1.50)

$$j\overrightarrow{k} \cdot \overrightarrow{E} = 0 \Rightarrow \overrightarrow{k} \perp \overrightarrow{E} \tag{1.55}$$

$$j\overrightarrow{k} \cdot \overrightarrow{H} = 0 \Rightarrow \overrightarrow{k} \perp \overrightarrow{B} \tag{1.56}$$

These both equations show that the fields $\overrightarrow{E}$ and $\overrightarrow{H}$ are transverse to the propagation direction defined along the vector $\overrightarrow{k}$. Since $\overrightarrow{E}(\overrightarrow{R},t) = \overrightarrow{E}_0 e^{-j(\omega t - \overrightarrow{k}\cdot\overrightarrow{r})}$ is solution of the Helmholtz equation, from Eq. (1.52), the wave number k verified the dispersion equation

$$-k^2 + (\epsilon\mu_0\omega^2 + j\mu_0\sigma\omega) = 0 \Rightarrow k = \sqrt{\epsilon\mu_0\omega^2 + j\mu_0\sigma\omega} \quad (1.57)$$

Introducing the refraction index n, the wave number k can be expressed as

$$k = \sqrt{\epsilon\mu_0\omega^2 + j\mu_0\sigma\omega} = \omega\sqrt{\epsilon_0\mu_0}\sqrt{\epsilon_r + j\frac{\sigma}{\omega\epsilon_0}} = k_0 \times n \quad (1.58)$$

where $k_0 = \omega\sqrt{\epsilon_0\mu_0}$ is the wave number in free space, for which $\epsilon_r = 1$ and $\sigma = 0$. In addition, the refraction index n is defined as

$$n = \sqrt{\epsilon_r + j\frac{\sigma}{\omega\epsilon_0}} = \sqrt{\epsilon_r}\sqrt{1 + j\frac{\sigma}{\omega\epsilon}} \quad (1.59)$$

We can notice that the refraction index n is a complex number and depends on the frequency. The medium is then called **dispersive**. By analogy, a complex relative permittivity can be defined as

$$\epsilon_{r1} = n^2 = \epsilon_r + j\frac{\sigma}{\omega\epsilon_0} = \epsilon_r + j\frac{18\sigma}{f} \text{ with } \left\{ \begin{array}{l} \sigma \text{ in S/m} \\ f \text{ in GHz} \end{array} \right. \quad (1.60)$$

From Eqs. (1.48), we have

$$j\overrightarrow{k} \wedge \overrightarrow{H} = -j\omega\epsilon\overrightarrow{E} + \sigma\overrightarrow{E} \Rightarrow \overrightarrow{H} \wedge \overrightarrow{k} = (\omega\epsilon + j\sigma)\,\overrightarrow{E} \quad (1.61)$$

which shows that $(\overrightarrow{E}, \overrightarrow{H}, \overrightarrow{k})$ are mutually transverse. In addition, from Eq. (1.61), we have

$$\begin{aligned} \left\|\overrightarrow{H}\right\|\left\|\overrightarrow{k}\right\|\underbrace{\sin\left(\widehat{\overrightarrow{H},\overrightarrow{k}}\right)}_{=1 \text{ why?}} = (\omega\epsilon + j\sigma)\left\|\overrightarrow{E}\right\| \Rightarrow \eta &= \frac{E}{H} = \frac{\left\|\overrightarrow{k}\right\|}{\omega\epsilon + j\sigma} \\ &= \frac{k_0 n}{\omega\epsilon + j\sigma} = \frac{\omega\sqrt{\epsilon_0\mu_0}n}{\omega\epsilon\left(1 + \frac{j\sigma}{\epsilon\omega}\right)} = \frac{\epsilon_r\sqrt{\epsilon_0\mu_0}n}{n^2\epsilon} \\ &= \sqrt{\frac{\mu_0}{\epsilon_0}}\frac{1}{n} = \frac{\eta_0}{n} \end{aligned} \quad (1.62)$$

where η is the wave **impedance** in ohm. The modulus of η gives the ratio modulus of E/H and the phase of η gives the phase difference between E and H. Unlike the vacuum, η is a complex number.

For example, for a plane wave propagating with respect to the direction $\overrightarrow{z}$ and polarized with respect to $\overrightarrow{x}$, $\overrightarrow{E}(z,t) = E_0\overrightarrow{x}e^{-j(\omega t - kz)}$ where $\overrightarrow{k} = k\overrightarrow{z}$. The magnetic field is then $\overrightarrow{H}(z,t) = E_0\overrightarrow{y}e^{-j(\omega t - kz)}/\eta$ or $\overrightarrow{H}(z,t) = (E_0/|\eta|)\overrightarrow{y}e^{-j(\omega t - kz - \phi)}$ where $\phi = \arg(\eta)$.

1.4 Plane wave reflection and transmission from a plane surface

This section is devoted to the calculation of the reflected and transmitted waves by a plane surface (of infinite area, which means that no diffraction phenomenon) illuminated by a plane wave.

As shown in figure 1.9, the upper medium 1 is defined for $z \geq 0$ of permittivity ϵ_1 and permeability $\mu_1 = \mu_0$, and the lower medium 2, is defined for $z < 0$ of permittivity ϵ_2 and permeability $\mu_2 = \mu_0$.

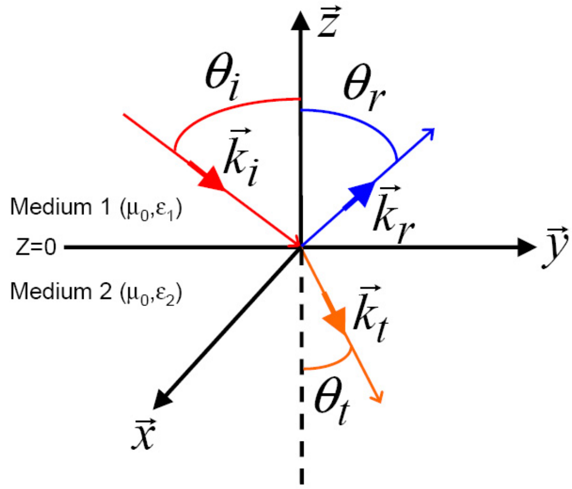

FIGURE 1.9 – The Snell-Descartes laws.

In general, for an infinite medium (no interface), an incident plane wave is expressed as $\overrightarrow{E}_i = \overrightarrow{E}_{0i} e^{-j(\omega_i t - \overrightarrow{k}_i \cdot \overrightarrow{r})}$, where the vector $\overrightarrow{E}_{0i}$ is related to the polarization and the amplitude of the wave. In addition, ω_i is the pulsation, $\overrightarrow{k}_i$ the wave vector, which gives the direction of the electric field, and $\overrightarrow{r}$ the vector position. All the variables $(\overrightarrow{E}_{0i}, \omega_i, \overrightarrow{k}_i)$ are known.

When we consider the problem shown in figure 1.9, the incident wave is reflected into the medium 1 and transmitted into the medium 2. For each medium, the Maxwell equations can be applied leading to that the reflected and transmitted fields can be written in a similar manner as the incident field. They are given by $\overrightarrow{E}_r = \overrightarrow{E}_{0r} e^{-j(\omega_r t - \overrightarrow{k}_r \cdot \overrightarrow{r})}$ and $\overrightarrow{E}_t = \overrightarrow{E}_{0t} e^{-j(\omega_t t - \overrightarrow{k}_t \cdot \overrightarrow{r})}$, respectively.

The problem to solve is to determine $(\overrightarrow{E}_{0r}, \omega_r, \overrightarrow{k}_r, \overrightarrow{E}_{0t}, \omega_t, \overrightarrow{k}_t)$. This problem is solved by applying the boundary conditions on the interface defined at $z = 0$.

1.4.1 Boundary conditions

Let S be a surface separating a medium 1 from a medium 2 and $\overrightarrow{n}$ the normal to the surface arbitrary oriented from 1 to 2. The boundary conditions at the interface ($z = 0$) are then expressed as

$$\overrightarrow{n} \wedge (\overrightarrow{E}_1 - \overrightarrow{E}_2) = \overrightarrow{0} \quad \text{Tangential component} \tag{1.63a}$$

$$\overrightarrow{n} \wedge (\overrightarrow{H}_1 - \overrightarrow{H}_2) = \overrightarrow{J}_S \quad \text{Tangential component} \tag{1.63b}$$

$$\overrightarrow{n} \cdot (\mu_1 \overrightarrow{H}_1 - \mu_2 \overrightarrow{H}_2) = 0 \quad \text{Normal component} \tag{1.63c}$$

$$\overrightarrow{n} \cdot (\epsilon_1 \overrightarrow{E}_1 - \epsilon_2 \overrightarrow{E}_2) = \rho_S \quad \text{Normal component} \tag{1.63d}$$

$\overrightarrow{J}_S$ is the current electric surface density and ρ_S is the charge electric surface density. We have then :

- **Continuity** of the **tangential** component of the electric field $\overrightarrow{E}$ and of the **normal** component of the magnetic field $\overrightarrow{H}$.
- **Discontinuity** of the **normal** component of the electric field $\overrightarrow{E}$ (due to the presence of ρ_S) and of the **tangential** component of the magnetic field $\overrightarrow{H}$ (due to the presence of $\overrightarrow{J}_S$).

If the media 2 is a **perfect conductor**[2], then Eq. (1.63) becomes

$$\overrightarrow{n} \wedge \overrightarrow{E}_1 = \overrightarrow{0} \quad \text{Tangential component} \tag{1.64a}$$

$$\overrightarrow{n} \wedge \overrightarrow{H}_1 = \overrightarrow{J}_S \quad \text{Tangential component} \tag{1.64b}$$

$$\overrightarrow{n} \cdot \overrightarrow{H}_1 = 0 \quad \text{Normal component} \tag{1.64c}$$

$$\overrightarrow{n} \cdot \overrightarrow{E}_1 = \rho_S/\epsilon_1 \quad \text{Normal component} \tag{1.64d}$$

If the media 1 and 2 are **perfect dielectric**, then $\overrightarrow{J}_S = \overrightarrow{0}$ and $\rho_S = 0$, leading from Eq. (1.63) to

$$\overrightarrow{n} \wedge (\overrightarrow{E}_1 - \overrightarrow{E}_2) = \overrightarrow{0} \quad \text{Tangential component} \tag{1.65a}$$

$$\overrightarrow{n} \wedge (\overrightarrow{H}_1 - \overrightarrow{H}_2) = \overrightarrow{0} \quad \text{Tangential component} \tag{1.65b}$$

$$\overrightarrow{n} \cdot (\mu_1 \overrightarrow{H}_1 - \mu_2 \overrightarrow{H}_2) = 0 \quad \text{Normal component} \tag{1.65c}$$

$$\overrightarrow{n} \cdot (\epsilon_1 \overrightarrow{E}_1 - \epsilon_2 \overrightarrow{E}_2) = 0 \quad \text{Normal component} \tag{1.65d}$$

1.4.2 Snell-Descartes laws

By applying the boundary conditions (continuity of the tangential component of the electric field), a relation between the amplitudes of the three waves (incident, reflected and transmitted) exist if for $z = 0$, the phase term of each exponential is equal. From

$$\begin{cases} \overrightarrow{E}_i = \overrightarrow{E}_{0i} e^{-j(\omega_i t - \overrightarrow{k}_i \cdot \overrightarrow{r})} \\ \overrightarrow{E}_r = \overrightarrow{E}_{0r} e^{-j(\omega_r t - \overrightarrow{k}_r \cdot \overrightarrow{r})} \\ \overrightarrow{E}_t = \overrightarrow{E}_{0t} e^{-j(\omega_t t - \overrightarrow{k}_t \cdot \overrightarrow{r})} \end{cases} \tag{1.66}$$

this leads to

$$\omega_i t - \overrightarrow{k}_i \cdot \overrightarrow{r} = \omega_r t - \overrightarrow{k}_r \cdot \overrightarrow{r} = \omega_t t - \overrightarrow{k}_t \cdot \overrightarrow{r} \quad \forall\, (\overrightarrow{r} \in S, t) \tag{1.67}$$

Thus

$$\omega_i t - k_{ix} x + k_{iy} y = \omega_r t - k_{rx} x + k_{ry} y = \omega_t t - k_{tx} x + k_{ty} y \tag{1.68}$$

with $\overrightarrow{r} = (x, y, z)$ and $\overrightarrow{k}_{i,r,t} = (k_{ix,rx,tx}, k_{iy,ry,ty}, k_{iz,rz,tz})$ since for $\overrightarrow{r} \in S$, $z = 0$. Noticing that the vector $\overrightarrow{k}_i$ lies in the (yOz) $(k_{ix} = 0)$ plane, $\forall\, (x, y, t)$, the above equation becomes

$$\begin{cases} \omega_i = \omega_r = \omega_t = \omega \\ k_{ix} = 0 = k_{rx} = k_{tx} \\ k_{iy} = k_{ry} = k_{ty} \end{cases} \tag{1.69}$$

2. A perfect conductor has a conductivity $\sigma \rightarrow j\infty$

Then

1. The first equation shows that the pulsations are **equal**.
2. The second equation shows that the incident, reflection and transmission planes, defined by the vectors $(\vec{z}, \vec{k}_i)$, $(\vec{z}, \vec{k}_r)$ and $(\vec{z}, \vec{k}_t)$, respectively, are the **same**.
3. From figure 1.9, the last equation shows that

$$k_i \sin\theta_i = k_r \sin\theta_r = k_t \sin\theta_t \tag{1.70}$$

Moreover, $k_i = k_r$ because the propagation medium is the same and $k_{i,t} = k_0 n_{1,2}$. Thus

$$\begin{cases} \theta_r = +\theta_i \\ n_1 \sin\theta_i = n_2 \sin\theta_t \end{cases} \tag{1.71}$$

They are the famous Snell-Descartes laws. The third one is the more famous but do not forget the others.

1.4.3 Fresnel coefficients

1.4.3.1 Case of a PC surface at a normal incidence for the TE polarization

For a perfectly conducting (PC) surface there is no transmitted field. In addition, we assume that the incidence angle is $\theta_i = 0$.

For the TE polarization, the incident electric field $\vec{E}_i$ is transverse, i.e., orthogonal to the incident plane or collinear to the vector $\vec{x}$.

By applying that $(\vec{E_i}, \vec{H_i}, \vec{k_i})$ (TEM structure of a plane wave) is an orthogonal direct basis, the direction of $\vec{H_i}$ is obtained (rule of the right hand).

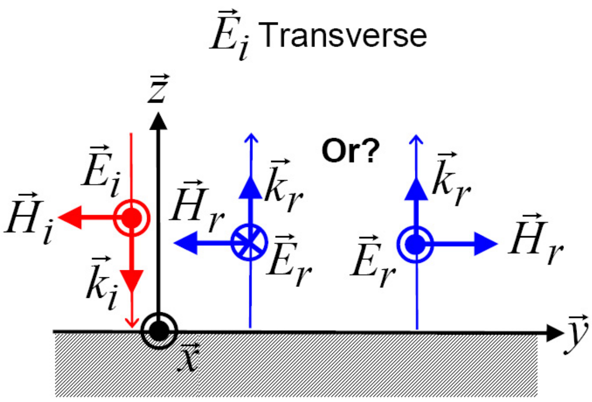

FIGURE 1.10 – Reflected elecromagnetic fields for a PC surface, the TE case and $\theta_i = 0$.

For $z = 0$, the boundary conditions (Eq. (1.64)) state that the tangential components of the total electric field vanishes on the interface S. Thus, since by construction, the vectors $\vec{E_i}$ and $\vec{E_r}$ are tangential to the surface, we have $\vec{E_i} + \vec{E_r} = \vec{0} \Rightarrow \vec{E_r} = -\vec{E_i}$. Thus, the direction of $\vec{E_r}$

$(-\overrightarrow{x})$ is opposite to that of $\overrightarrow{E_i}$ $(+\overrightarrow{x})$. In addition, since $(\overrightarrow{E_r}, \overrightarrow{H_r}, \overrightarrow{k_r})$ (TEM structure of a plane wave) is an orthogonal direct basis, the vectors $\overrightarrow{H_i}$ and $\overrightarrow{H_r}$ are in the same direction.

Thus, the reflection and transmission coefficients are

$$\mathcal{R}_H = \frac{E_{0r}}{E_{0i}} = -1 \quad \mathcal{T}_H = \frac{E_{0t}}{E_{0i}} = 0 \tag{1.72}$$

For the TE polarization, the subscript H is used as horizontal.

The total electric and magnetic fields in the medium 1 are then

$$\begin{cases} \overrightarrow{E}_t = \overrightarrow{E}_i + \overrightarrow{E}_r = \overrightarrow{x} E_{0i} e^{-j\omega t} \left(e^{-jk_1 z} - e^{jk_1 z}\right) = -2j \overrightarrow{x} E_{0i} e^{-j\omega t} \sin(k_1 z) \\ \overrightarrow{H}_t = \overrightarrow{H}_i + \overrightarrow{H}_r = -\overrightarrow{y} H_{0i} e^{-j\omega t} \left(e^{-jk_1 z} + e^{jk_1 z}\right) = -2 \overrightarrow{y} H_{0i} e^{-j\omega t} \cos(k_1 z) \end{cases} \tag{1.73}$$

In addition, $H_{0i} = E_{0i}/\eta_1$, where η_1 is the wave impedance of the medium 1. We can also note that $k_1 = k_0 n_1$, where n_1 is the refraction index of the medium 1.

1.4.3.2 Case of a dielectric surface at a normal incidence for the TE polarization

Now, we consider that the medium 2 is a perfect dielectric. Thus, a transmitted field can be propagated in the medium 2.

From figure 1.10 (by convention, we use the picture on the right) and applying the boundary conditions at $z = 0$ (Eqs. (1.65) on the tangential components), we have

$$\begin{cases} E_{0i} + E_{0r} = E_{0t} \\ -H_{0i} + H_{0r} = -H_{0t} \end{cases} \tag{1.74}$$

In addition, $H_{0i} = E_{0i}/\eta_1 = n_1 E_{0i}/\eta_0$, $H_{0r} = E_{0r}/\eta_1 = n_1 E_{0r}/\eta_0$ and $H_{0t} = E_{0t}/\eta_2 = n_2 E_{0t}/\eta_0$. Thus

$$\begin{cases} E_{0r} + E_{0i} = E_{0t} \\ E_{0i} - E_{0r} = \frac{n_2}{n_1} E_{0t} \end{cases} \tag{1.75}$$

In conclusion

$$\begin{cases} \mathcal{R}_H = \dfrac{E_{0r}}{E_{0i}} = \dfrac{n_1 - n_2}{n_1 + n_2} \\ \\ \mathcal{T}_H = \dfrac{E_{0t}}{E_{0i}} = \dfrac{2n_1}{n_1 + n_2} \end{cases} \tag{1.76}$$

For a PC surface, $|n_2| \to \infty$, then $\mathcal{R}_H = -1$ and Eq. (1.72) is retrieved.

1.4.3.3 Case of a dielectric surface for the TE polarization

In this subsection, the general case of a perfect dielectric surface is considered for the TE polarization.

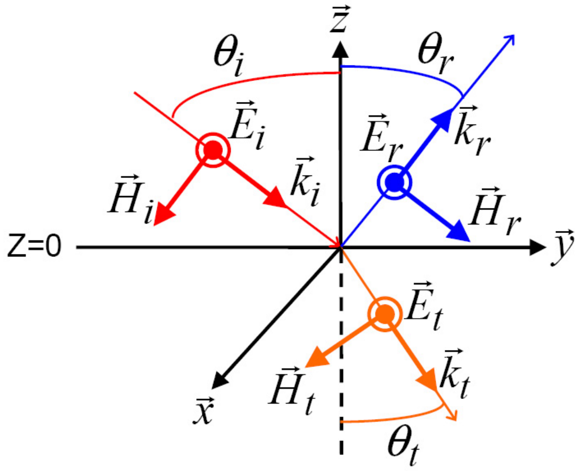

FIGURE 1.11 – Fresnel coefficients for the TE polarisation and a perfect dielectric medium.

By applying that $(\overrightarrow{E_i}, \overrightarrow{H_i}, \overrightarrow{k_i})$ is an orthogonal direct basis, the direction of $\overrightarrow{H_i}$ is obtained (rule of the right hand). As shown in figure 1.11, the same way is used for $(\overrightarrow{E_r}, \overrightarrow{H_r}, \overrightarrow{k_r})$ and $(\overrightarrow{E_t}, \overrightarrow{H_t}, \overrightarrow{k_t})$.

From Eqs. (1.65), the tangential components of the electric and magnetic fields are continuous on the interface S defined at $z = 0$. From figure 1.11, this leads for $\forall\ (x, y)$ to

$$\begin{cases} E_{0i} + E_{0r} = E_{0t} \\ -H_{0i}\cos\theta_i + H_{0r}\cos\theta_r = -H_{0t}\cos\theta_t \end{cases} \tag{1.77}$$

Moreover, from Eq. (1.62), $H_{0i} = n_1 E_{0i}/\eta_0$, $H_{0r} = n_1 E_{0r}/\eta_0$ and $H_{0t} = n_2 E_{0t}/\eta_0$, leading with $\theta_i = \theta_r$ to

$$\begin{cases} E_{0i} + E_{0r} = E_{0t} \\ E_{0i} - E_{0r} = \dfrac{n_2}{n_1}\dfrac{\cos\theta_t}{\cos\theta_i} E_{0t} \end{cases} \tag{1.78}$$

Letting $\mathcal{R}_H = E_{0r}/E_{0i}$ (reflection coefficient) and $\mathcal{T}_H = E_{0t}/E_{0i}$ (transmission coefficient), we obtain

$$\begin{cases} \mathcal{R}_H = \dfrac{n_1\cos\theta_i - n_2\cos\theta_t}{n_1\cos\theta_i + n_2\cos\theta_t} = \dfrac{n_1\cos\theta_i - \sqrt{n_2^2 - n_1^2\sin^2\theta_i}}{n_1\cos\theta_i + \sqrt{n_2^2 - n_1^2\sin^2\theta_i}} \\ \mathcal{T}_H = 1 + \mathcal{R}_H = \dfrac{2n_1\cos\theta_i}{n_1\cos\theta_i + n_2\cos\theta_t} \end{cases} \tag{1.79}$$

where the third Snell-Descartes law $n_1 \sin\theta_i = n_2 \sin\theta_t$ is used. For $\theta_i = 0$, Eq. (1.76) is retrieved.

1.4.3.4 Case of a dielectric surface for the TM polarization

In this subsection, the general case of a perfect dielectric surface is considered for the TM polarization (figure 1.12).

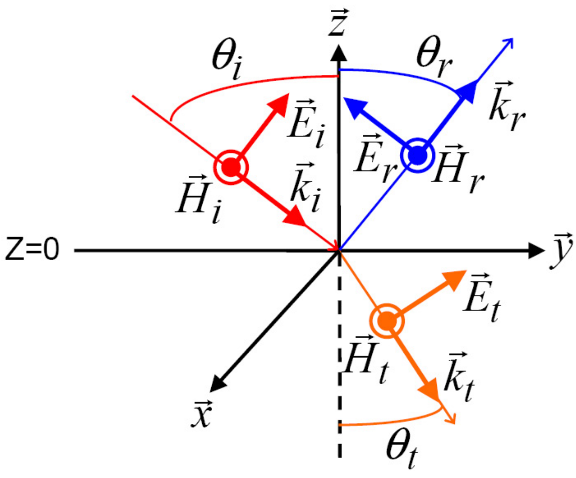

FIGURE 1.12 – Fresnel coefficients for the TM polarization and a perfect-dielectric medium.

From Eqs. (1.65), the tangential components of the electric and magnetic fields are continuous on the interface S defined at $z = 0$. From figure 1.11, this leads for $\forall\ (x, y)$ to

$$\begin{cases} H_{0i} + H_{0r} = H_{0t} \\ E_{0i}\cos\theta_i - E_{0r}\cos\theta_r = E_{0t}\cos\theta_t \end{cases} \tag{1.80}$$

Thus

$$\begin{cases} E_{0i} + E_{0r} = \dfrac{\sin\theta_i}{\sin\theta_t} E_{0t} \\ \\ E_{0i} - E_{0r} = \dfrac{\cos\theta_t}{\cos\theta_i} E_{0t} \end{cases} \tag{1.81}$$

Letting $\mathcal{R}_V = E_{0r}/E_{0i}$ (reflection coefficient) and $\mathcal{T}_V = E_{0t}/E_{0i}$ (transmission coefficient), we obtain

$$\begin{cases} \mathcal{R}_V = \dfrac{n_2\cos\theta_i - n_1\cos\theta_t}{n_2\cos\theta_i + n_1\cos\theta_t} = \dfrac{n_2^2\cos\theta_i - n_1\sqrt{n_2^2 - n_1^2\sin^2\theta_i}}{n_2^2\cos\theta_i + n_1\sqrt{n_2^2 - n_1^2\sin^2\theta_i}} \\ \mathcal{T}_V = \dfrac{n_1}{n_2}(1 + \mathcal{R}_V) = \dfrac{2n_1\cos\theta_i}{n_2\cos\theta_i + n_1\cos\theta_t} \end{cases} \tag{1.82}$$

where the third Snell-Descartes law $n_1\sin\theta_i = n_2\sin\theta_t$ is used.

For the TM polarization, the subscript V is used as vertical.

1.4.3.5 Discussion on the Fresnel formula

For θ_i close to zero, $\sin\theta_i \approx \theta_i$ and $\sin\theta_t \approx n_1\theta_i/n_2 \approx \theta_t$. Thus, from Eqs. (1.79) and (1.82), the Fresnel coefficients can be simplified as

$$\begin{cases} \mathcal{R}_H = \dfrac{\theta_t - \theta_i}{\theta_t + \theta_i} \approx \dfrac{n_1 - n_2}{n_1 + n_2} \\ \mathcal{R}_V \approx -\mathcal{R}_H \end{cases} \tag{1.83}$$

In the air-($n_1 = 1$)-glass ($n_2 = 1.5$), $\mathcal{R}_H = -0.2$ et $\mathcal{R}_V = 0.2$. This means that for the TE polarization, the reflected electric field is in opposite sense because $\mathcal{R}_H < 0$.

For grazing incidences, $\theta_i = \pi/2$, this leads from Eqs. (1.79) and (1.82) to

$$\mathcal{R}_H \approx \mathcal{R}_V \approx -1 \tag{1.84}$$

Figures 1.13 and 1.14 plot the Fresnel coefficients in reflexion and transmission with respect to the polarizations TM ($\mathcal{R}_V$,$\mathcal{T}_V$) and TE ($\mathcal{R}_H$, $\mathcal{T}_H$) and for an air-glass interface.

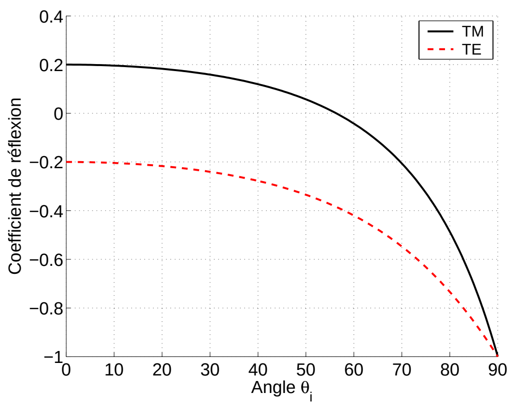

FIGURE 1.13 – Reflexion coefficients for TE and TM polarizations, $n_1 = 1$ and $n_2 = 1.5$.

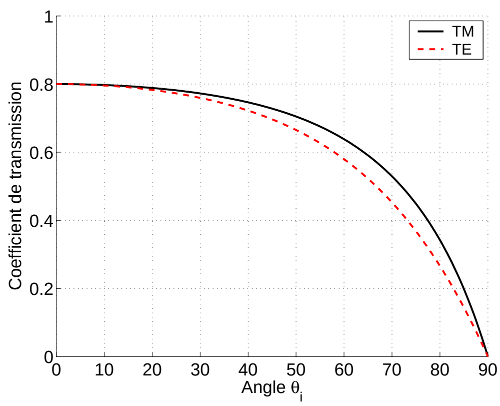

FIGURE 1.14 – Transmission coefficients for TE and TM polarizations, $n_1 = 1$ and $n_2 = 1.5$

For the TM polarization, we observe that $\mathcal{R}_V$ reaches zero. From Eq. (1.79), this angle satisfied $\theta_{iB} + \theta_{tB} = \pi/2$ (numerator equal zero and $n_1 \sin\theta_i = n_2 \sin\theta_t$), implying that $\theta_{tB} = \pi/2 - \theta_{iB}$. Moreover, $n_1 \sin\theta_{iB} = n_2 \sin(\pi/2 - \theta_{iB}) = n_2 \cos\theta_{iB}$. Thus

$$\tan\theta_{iB} = n_2/n_1 \tag{1.85}$$

θ_{iB} is called the **Brewster angle**. For an air-glass interface, $\theta_{iB} = 56.3$ degrees. For this particular value, $\mathcal{T}_V(\theta_{iB}) \neq 0$, $\mathcal{T}_H(\theta_{iB}) \neq 0$ and $\mathcal{R}_H(\theta_{iB}) \neq 0$, whereas $\mathcal{R}_V(\theta_{iB}) = 0$. This property is then used for optics sensors to generate particular polarization states.

If the numbers n_1 or/and n_2 are complex, the Fresnel coefficients are also complex.

If $n_1 > n_2$, a limit incidence angle, θ_{iL}, can be calculated for which the transmission angle equals $\theta_t = \pi/2$. This implies that $\sin\theta_{iL} = n_2/n_1 \leq 1$. For a glass-air interface, it is equal to 42 degrees. As shown in figures 1.15-1.18, above this angle, the Fresnel coefficients give complex values.

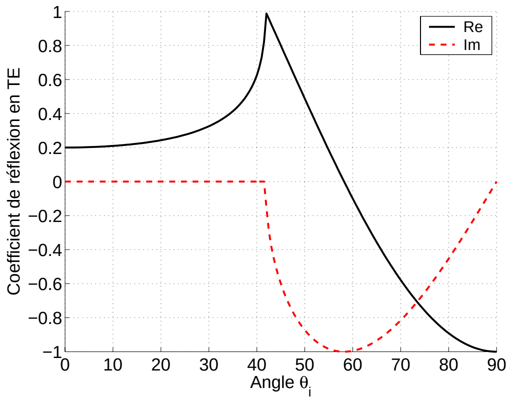

FIGURE 1.15 – Real and imaginary parts of the **reflection** coefficient for the TE polarization, $n_1 = 1.5$ and $n_2 = 1$.

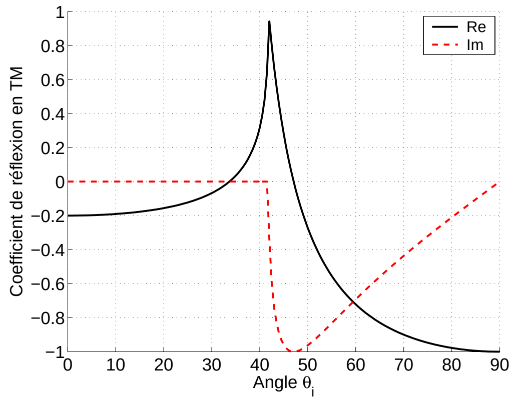

FIGURE 1.16 – Real and imaginary parts of the **reflection** coefficient for the TM polarization, $n_1 = 1.5$ and $n_2 = 1$.

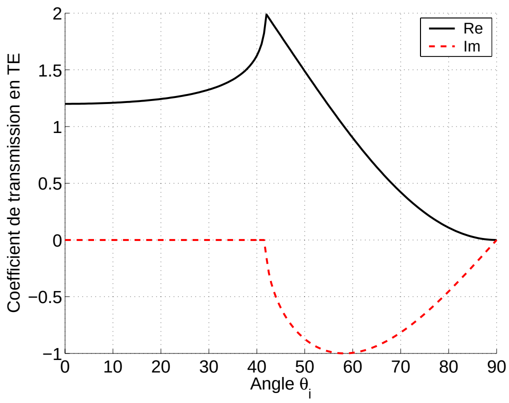

FIGURE 1.17 – Real and imaginary parts of the **transmission** coefficient for the TE polarization, $n_1 = 1.5$ and $n_2 = 1$.

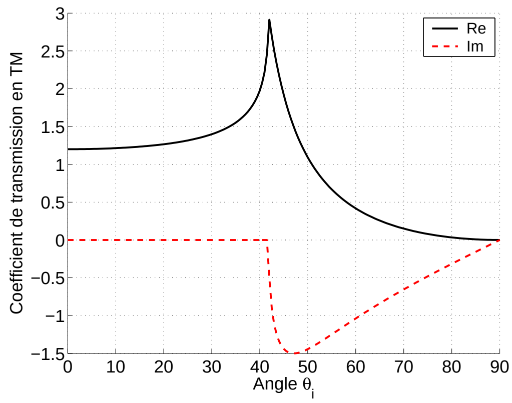

FIGURE 1.18 – Real and imaginary parts of the **transmission** coefficient for the TM polarization, $n_1 = 1.5$ and $n_2 = 1$.

1.5 Exercises

1.5.1 Exercises on the Fresnel coefficients

1.5.1.1 Exercise 1

We consider an interface of infinite area lied in the plane $(\overrightarrow{x}, \overrightarrow{y})$ separating two LHI media. The upper medium, defined for $z \geq 0$, is vacuum and the lower medium, defined for $z < 0$, is a perfect dielectric medium of complex refraction index $n = n_r + jn_i$ $((n_r, n_i) \in \mathbb{R}^+)$. The interface is illuminated by a TEM plane wave $\overrightarrow{E}_i$ polarized along the direction $\overrightarrow{x}$ and propagating along the $\overrightarrow{z}$ direction $(\overrightarrow{k}_i = k_i \overrightarrow{z})$. Thus, $\overrightarrow{E}_i = E_{0i} e^{-j(\omega_i t - k_i z)} \overrightarrow{x}$.

1. Do a figure of the problem.
2. Give the polarization of the incident wave ?

3. Express the incident wave number k_i from the wavelength λ_0 in the vacuum.
4. Simplified then the expression of $\overrightarrow{E}_i$.

The transmitted electric field is $\overrightarrow{E}_t = E_{0t} e^{-j(\omega_t t - \overrightarrow{k}_t \cdot \overrightarrow{r})} \overrightarrow{p}_t$.

1. Give the polarization $\overrightarrow{p}_t$ of the transmitted electric field.
2. Give the relation between ω_t and ω_i.
3. Express $\overrightarrow{k}_t$ from $\{k_0, n, \overrightarrow{x}, \overrightarrow{y}, \overrightarrow{z}\}$.
4. Express E_{0t} from E_{0i} and the transmission coefficient $\mathcal{T}$ and next n.
5. Express then $\overrightarrow{E}_t(z)$ from $\{k_0, n, E_{0i}, \omega_i, \overrightarrow{x}, \overrightarrow{y}, \overrightarrow{z}\}$.
6. Calculate $\rho(z) = \left\|\overrightarrow{E}_t(z)\right\| / \left\|\overrightarrow{E}_t(0)\right\|$ and $|\rho(z)|$.
7. Calculate the skin depth $z = \delta$, for which $|\rho(z)| = e^{-1}$. Conclude.

1.5.1.2 Exercise 2

In LHI conductor medium, the Maxwell equations are given by

$$\overrightarrow{\nabla} \wedge \overrightarrow{H} = \frac{\partial \overrightarrow{D}}{\partial t} + \overrightarrow{J}$$

$$\overrightarrow{\nabla} \wedge \overrightarrow{E} = -\frac{\partial \overrightarrow{B}}{\partial t}$$

$$\overrightarrow{\nabla} \cdot \overrightarrow{B} = 0$$

$$\overrightarrow{\nabla} \cdot \overrightarrow{D} = \rho$$

where

$$\overrightarrow{D} = \epsilon \overrightarrow{E}$$
$$\overrightarrow{B} = \mu_0 \overrightarrow{H}$$
$$\overrightarrow{J} = \sigma \overrightarrow{E}$$

1. Give the names of ϵ, μ_0 and σ and their unity.
2. We assume that $\rho = 0$. Show that the wave propagation is expressed as :

   $$\overrightarrow{\nabla}^2 \overrightarrow{E} - \mu_0 \sigma \frac{\partial \overrightarrow{E}}{\partial t} - \epsilon_0 \mu_0 \frac{\partial^2 \overrightarrow{E}}{\partial t^2} = \overrightarrow{0}$$

   You can use the identity $\overrightarrow{\nabla} \wedge (\overrightarrow{\nabla} \wedge \overrightarrow{A}) = -\overrightarrow{\nabla}^2 \overrightarrow{A} + \overrightarrow{\nabla}(\overrightarrow{\nabla} \cdot \overrightarrow{A})$ for any vector $\overrightarrow{A}$.
3. We assume that $\overrightarrow{E}(\overrightarrow{r}, t) = \overrightarrow{E}_0(\overrightarrow{r}) e^{-j\omega t}$. Show then

   $$(\overrightarrow{\nabla}^2 + k_0^2 n^2) \overrightarrow{E}_0(\overrightarrow{r}) = \overrightarrow{0}$$

   where $k_0 = \omega / c$, in which $c = 1/\sqrt{\epsilon_0 \mu_0}$ is the wave speed in vacuum. Give the expression of n and its name.

1.5.1.3 Exercise 3

The power carried by an electric field propagating in a medium with **lossy** (means that the refractive index $n \in \mathbb{C}$), is defined from the Poynting vector $\overrightarrow{P}$ as

$$\overrightarrow{P} = \frac{1}{2}\overrightarrow{E} \wedge \overrightarrow{H}^* \tag{E1}$$

where $\overrightarrow{H}$ is the magnetic field and the symbol * is the conjugate. The refractive index $n = n_r + jn_i$ with $(n_i, n_r) \in \mathbb{R}^+$. We assume that the electric field is expressed as $\overrightarrow{E} = E_0 \overrightarrow{x} e^{jn\overrightarrow{k}_0 \cdot \overrightarrow{r}}$, where $\overrightarrow{k}_0$ is the wave vector in the vacuum. An $e^{-j\omega t}$ time dependence is assumed.

1. From a Maxwell equation, calculate $\overrightarrow{H}$.
2. Show then that $\overrightarrow{P} = \frac{n^*}{2\eta_0}\left|\overrightarrow{E}\right|^2 \overrightarrow{u}$ with $\overrightarrow{u} = \overrightarrow{k}_0/k_0$ (unitary vector). η_0 is the wave impedance in the vaccum. For any vectors $(\overrightarrow{V}_1, \overrightarrow{V}_2, \overrightarrow{V}_3)$ we have
$$\overrightarrow{V}_1 \wedge (\overrightarrow{V}_2 \wedge \overrightarrow{V}_3) = (\overrightarrow{V}_1 \cdot \overrightarrow{V}_3)\overrightarrow{V}_2 - (\overrightarrow{V}_1 \cdot \overrightarrow{V}_2)\overrightarrow{V}_3. \tag{E2}$$
3. We set $\overrightarrow{k}_0 = k_0 \overrightarrow{z}$. Express then $\overrightarrow{P}(z)$ versus z.
4. Calculate $\rho(z) = \left\|\overrightarrow{P}(z)\right\| / \left\|\overrightarrow{P}(0)\right\|$.
5. Plot $\rho(z)$ versus z and conclude.

1.5.1.4 Exercise 4

We consider an incident TEM plane wave which illuminates two infinite interfaces Σ_A and Σ_B separating LHI media $\{\Omega_1, \Omega_2, \Omega_3\}$ of refractive indexes n_1 (assumed to be the air), n_2 and n_3. The polarisation of the incident plane wave is TE with an incidence angle $\theta_i = 0$ ($\overrightarrow{E} = \psi^i e^{jk_1 z}\overrightarrow{x}$). The Fresnel coefficients in reflection and transmission from the medium $i = \{1, 2, 3\}$ to the medium $j \neq i = \{1, 2, 3\}$ are denoted as r_{ij} and t_{ij}, respectively (figure 1.19).

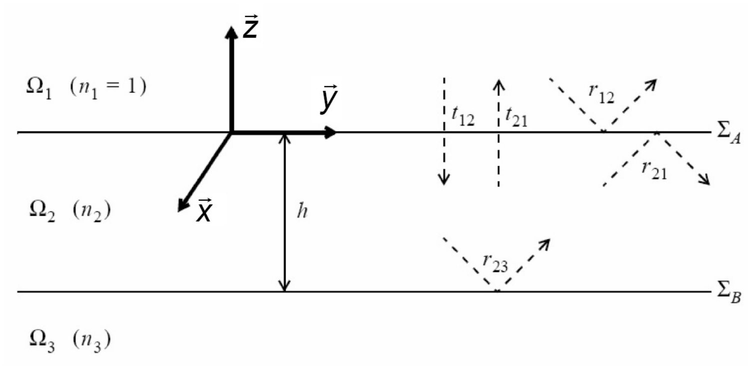

FIGURE 1.19 – Description of the geometry.

1. From a figure, explain qualitatively that the magnitude of the reflected field ψ^r can be written as follows :
$$\psi^r = \sum_{p=0}^{p=\infty} \psi^r_p. \tag{E3}$$

2. Give the expressions of r_{12}, r_{21}, t_{12} and t_{21} and show that $t_{12}t_{21} = 1 - r_{12}^2$.
3. Give the expression of the field ψ_0^r reflected by *only* the upper interface Σ_A.
4. Give the expression of the field ψ_1^r for $p = 1$. It results from the transmission through the upper interface Σ_A, the reflection from the lower interface Σ_B, and then the transmission through Σ_A back into the incident medium Ω_1.
5. Show that the reflected field at the order $p = 2$ is

$$\psi_2^r = \left(r_{21}r_{23}e^{j\phi}\right)\psi_1^r, \tag{E4}$$

where $\phi = 2k_0n_2h$, in which k_0 is the wavenumber in the air (vaccum) and h the thickness of the intermediate medium Ω_2.
6. Show that the reflected field at the order $p \geq 1$, is then

$$\psi_p^r = \left(r_{21}r_{23}e^{j\phi}\right)^{p-1}\psi_1^r. \tag{E5}$$

7. From equation (E3), and the relations $t_{12}t_{21} = 1 - r_{12}^2$ and $r_{21} = -r_{12}$, show that the total reflected field is expressed as

$$\psi^r = \psi^i \frac{r_{12} + r_{23}e^{j\phi}}{1 + r_{12}r_{23}e^{j\phi}}. \tag{E6}$$

We recall for $|x| < 1$ that $\sum_{p=1}^{p=\infty} x^{p-1} = \frac{1}{1-x}$.
8. We assume now that the medium Ω_3 is **perfectly conducting**. Give the value of r_{23} and simplify equation (E6).
9. Moreover, we assume that the modulus of the refractive index n_2 is of the order of n_1 ($|n_2| \approx |n_1|$). Give the consequence on $|r_{12}|$ and show that

$$\psi^r \approx \left[r_{12}\left(1 - e^{j2\phi}\right) + e^{j\phi}\right]\psi^i. \tag{E7}$$

We recall for $x \to 0$ that $1/(1+x) = 1 - x + x^2 + \mathcal{O}(x^2)$.

1.5.2 Exercices on the polarization

1.5.2.1 Excercise 1

From Eq. (1.45), show Eq. (1.46).

1.5.2.2 Excercise 2

In this exercise, we want to retreive Eqs. (1.47).

In the basis $(x = E_x, y = E_y)$ (top of figure 1.8), the equation of the ellipse is given from Eq. (1.42). When the ellipse undergone a rotation of α (bottom of figure 1.8), its equation is expressed from Eq. (1.46). In addition, in the basis (x', y'), the equation of the same ellipse is

$$\frac{x'^2}{a'^2} + \frac{y'^2}{b'^2} = 1 \tag{E8}$$

The couple (x, y) is expressed from (x', y') by a rotation of an angle $-\alpha$. Then

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix} \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x'\cos\alpha - y'\sin\alpha \\ x'\sin\alpha + y'\cos\alpha \end{bmatrix} \tag{E9}$$

1. Reporting Eq. (E9) into Eq. (1.46) and equaling Eq. (1.46) with Eq. (E8), show that the term with respect to $x'y'$ vanishes if

$$\tan(2\alpha) = \frac{2ab\cos\delta}{a^2 - b^2} \tag{E10}$$

   Note that $\sin(2\alpha) = 2\cos\alpha\sin\alpha$ and $\cos(2\alpha) = \cos^2\alpha - \sin^2\alpha$.

2. Reporting Eq. (E9) into Eq. (1.46) and equaling Eq. (1.46) with Eq. (E8), show that (terms with respect to x'^2 and y'^2)

$$a'b' = ab\sin\delta \tag{E11}$$

3. From Eq. (E9) show that

$$a^2 + b^2 = a'^2 + b'^2 \tag{E12}$$

4. Writting that $\sin(2\chi) = 2\tan\chi/(1 + \tan^2\chi)$ with $\tan\chi = b'/a'$, show that

$$\sin(2\chi) = \frac{2ab\sin\delta}{a^2 + b^2} \tag{E13}$$

1.5.2.3 Excercice 3

Fill the following table and locate the state polarization on the Poincaré Sphere.

E_{0x}	E_{0y}	δ	$\vec{S}$	Name of the polarization	α	χ
1	0	0				
0	1	0				
1	1	0				
1	-1	0				
1	1	$\pi/2$				
1	1	$-\pi/2$				

TABLE 1.2 – Fill the table.

2 Basic concepts on the propagation

2.1 Radiation from a point source

2.1.1 Spherical wave

Solving the Maxwell equation, we can show for a two-dimensional problem (2D problem, i.e. invariant along an arbitrary direction, for example $\overrightarrow{x}$) that the wave is **cylindrical**, which means that the electric field behaves as $1/\sqrt{R}$, where R is the distance between the sensor and the emitter. For a 3D problem (problems meet in the nature), the electric field behaves as $1/R$ and the wave is then **spherical**.

As shown in figure 2.1, if R is great, locally, the amplitude of the electric field measures by the receiver can be considered as a constant since $R_1 \approx R_2$. Then, the wave can be considered as **locally** plane.

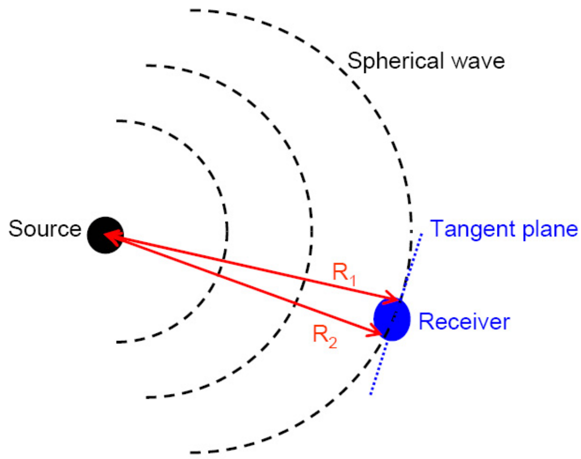

FIGURE 2.1 – Illustration of a spherical wave.

2.1.2 Radiated power

If P_0 (in W) is the power radiated by an isotropic source, for a spherical wave, the power density p_0 (in W/m^2) at the distance R is

$$p_0 = \frac{P_0}{4\pi R^2}, \tag{2.1}$$

where $4\pi R^2$ is the area of a sphere of radius R.

For a TEM plane wave, the power density carried by the wave is related from the Poynting vector $\overrightarrow{P}$ defined as

$$\overrightarrow{P} = \frac{1}{2}\overrightarrow{E} \wedge \overrightarrow{H}^* \tag{2.2}$$

For a plane wave propagated in free space, we have $\overrightarrow{E} = \overrightarrow{E}_0 e^{-j\omega t + j\overrightarrow{k}_0 \cdot \overrightarrow{R}}$. Thus, from a Maxwell equation, we have

$$\begin{aligned}
&\overrightarrow{\mathrm{rot}}\overrightarrow{E} = -\frac{\partial \overrightarrow{B}}{\partial t} = j\omega\mu_0\overrightarrow{H} \\
&\Rightarrow \overrightarrow{H} = \frac{1}{j\omega\mu_0}\overrightarrow{\nabla} \wedge \overrightarrow{E} = \frac{1}{\omega\mu_0}\overrightarrow{k_0} \wedge \overrightarrow{E} \\
&\Rightarrow \overrightarrow{P} = \frac{1}{2\omega\mu_0}\overrightarrow{E} \wedge \left(\overrightarrow{k}_0 \wedge \overrightarrow{E}\right)^* = \frac{1}{2\omega\mu_0}\overrightarrow{E} \wedge \left(\overrightarrow{k}_0 \wedge \overrightarrow{E}^*\right) \\
&\Rightarrow \overrightarrow{P} = \frac{1}{2\omega\mu_0}\left[\left(\overrightarrow{E} \cdot \overrightarrow{E}^*\right)\overrightarrow{k}_0 - \left(\underbrace{\overrightarrow{E} \cdot \overrightarrow{k}_0}_{=0}\right)\overrightarrow{E}^*\right] = \frac{|E|^2}{2\omega\mu_0}\overrightarrow{k}_0 \\
&\Rightarrow \overrightarrow{P} = \frac{|E_0|^2}{2Z_0}\overrightarrow{u}
\end{aligned} \tag{2.3}$$

where $k_0 = \omega/c$, $\overrightarrow{u} = \overrightarrow{k}_0/\left\|\overrightarrow{k}_0\right\|$ and $Z_0 = \sqrt{\mu_0/\epsilon_0} = 120\pi$. We note that the wave power is propagated along the direction $\overrightarrow{k}_0$ (direction of light rays).

Then

$$p_0 = \left\|\overrightarrow{P}\right\| = \frac{|E_0|^2}{2Z_0} \tag{2.4}$$

Then, the elctric field $\overrightarrow{E}$ is related to the power P_0 from

$$\overrightarrow{E} = \frac{\sqrt{60P_0}}{R}e^{-j\omega t + j\overrightarrow{k}_0 \cdot \overrightarrow{R}}\overrightarrow{u} \tag{2.5}$$

The electric field $\overrightarrow{E}$ behaves as $1/R$ which corresponds to a spherical wave.

2.1.3 Electric field calculation in presence of a ground

As shown in figure 2.2, we want to calculate the received field in presence of a ground.

From Eq. (2.5), the norm of the incident field is given by

$$E_1 = \frac{\sqrt{60P_1}}{R}e^{jk_0R} \tag{2.6}$$

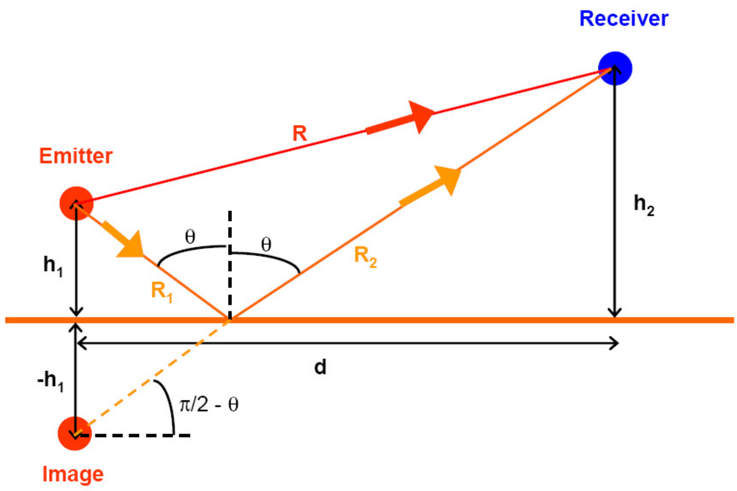

FIGURE 2.2 – Electric field calculation in presence of a ground.

where the depedence over the time t is omitted.

From the image theory, the norm of the relfected field E_2 by the ground can be replaced by a source located at the height $-h_1$ and of amplitude $\mathcal{R}(\theta)$. Thus

$$E_2 = \frac{\sqrt{60P_1}}{R_1 + R_2} e^{jk_0(R_1+R_2)} \mathcal{R}(\theta) \tag{2.7}$$

Then, the total field is

$$\begin{aligned} E &= E_1 + E_2 = \sqrt{60P_1}\left[\frac{e^{jk_0R}}{R} + \mathcal{R}\frac{e^{jk_0(R_1+R_2)}}{R_1+R_2}\right] \\ &= \frac{\sqrt{60P_1}}{R} e^{jk_0R}\left[1 + \mathcal{R}\frac{Re^{jk_0(R_1+R_2-R)}}{R_1+R_2}\right] \\ &= E_1\left[1 + \mathcal{R}\frac{Re^{jk_0(R_1+R_2-R)}}{R_1+R_2}\right] \end{aligned} \tag{2.8}$$

From the Pythagore theorem, we have

$$\begin{cases} R_1 + R_2 = \sqrt{d^2 + (h_1+h_2)^2} = d\sqrt{1 + \left(\frac{h_1+h_2}{d}\right)^2} \approx d\left[1 + \frac{1}{2}\left(\frac{h_1+h_2}{d}\right)^2\right] \\ R = \sqrt{d^2 + (h_1-h_2)^2} = d\sqrt{1 + \left(\frac{h_1-h_2}{d}\right)^2} \approx d\left[1 + \frac{1}{2}\left(\frac{h_1-h_2}{d}\right)^2\right] \end{cases} \tag{2.9}$$

where $d > 0$ is the horizontal distance between the emitter and the receicer, which is assumed to be much greater than the heights h_1 and h_2 of the emitter and receiver, respectively.

Thus

$$R_1 + R_2 - R \approx \frac{2h_1h_2}{d} \qquad \frac{R}{R_1+R_2} \approx 1 \tag{2.10}$$

The total field can then be approximated by

$$E = E_1 \left[1 + \mathcal{R}(\theta)e^{j\phi}\right] \tag{2.11}$$

where

$$\phi = \frac{2k_0 h_1 h_2}{d} = \frac{4\pi h_1 h_2}{\lambda_0 d} \qquad \tan\theta = \frac{d}{h_1 + h_2} \tag{2.12}$$

The modulus ratio of the electric field is then

$$p = \left|\frac{E}{E_1}\right| = \sqrt{1 + 2a\cos\phi' + a^2} \tag{2.13}$$

where

$$a = |\mathcal{R}| \qquad \phi' = \phi + \mathrm{phase}(\mathcal{R}) = \phi + \phi_{\mathcal{R}} \tag{2.14}$$

The minimum value of p occurs ($\cos\phi' = -1$) for $\phi' = \pi + 2n\pi$ (with n an integer). The maximum value of p occurs ($\cos\phi' = +1$) for $\phi' = 2n\pi$. This leads to

$$p_{\mathrm{max}} = |1 + a| \qquad p_{\mathrm{min}} = |1 - a| \tag{2.15}$$

For example, for a perfectly-conducting surface, $\mathcal{R} = \pm 1$, implying that $a = 1$ and then $p_{\mathrm{min}} = 0$. This means that the total field vanished. In pratice, this phenomenon is constraining because the communication is broken. In opposite, $p_{\mathrm{max}} = 2$ and then the total field is equal twice the emitter field. It is an illustration of the **inteference** phenomenon : **"1+1" can give "0"** !

For $h_1 = \mathrm{cste}$, $d = \mathrm{cste}$ and h_2 varies, the periodicity Δh_2 of h_2 satisfied

$$\frac{4\pi h_1 h_2}{\lambda_0 d} + \phi_{\mathcal{R}} = 2\pi \Rightarrow \Delta h_2 = \frac{\lambda_0 d}{2h_1} \tag{2.16}$$

For $h_1 = \mathrm{cste}$, $h_2 = \mathrm{cste}$ and d varies, the periodicity Δd of d satisfied

$$\frac{4\pi h_1 h_2}{\lambda_0 d} + \phi_{\mathcal{R}} = 2\pi \Rightarrow \Delta d = \frac{\lambda_0 d^2}{2h_1 h_2} \tag{2.17}$$

For the simulations, we assume that $\mathcal{R} = +1$, coressponding to a perfectly-conducting surface and the TE polarization. Thus, $a = 1$ and $\phi_{\mathcal{R}} = 0$. In addition, the frequency is $f = 300$ MHz.

Figure 2.3 plots p (Eq. (2.13)) versus the receiver height h_2 for $h_1 = 50$ m and $d = 10$ km. For this case, from Eq. (2.16), $\Delta h_2 = 100$ m. As we can see, p is a periodic function of h_2 of period Δh_2 and takes values from 0 to 2, as predicted from Eq. (2.15).

Figure 2.4 plots p (Eq. (2.13)) versus the horizontal distance d for $h_1 = 100$ m and $h_2 = 200$. For this case, from Eq. (2.17), Δd is not a constant and varies with d. As we can see, p is not a periodic function of d and takes values from 0 to 2, as predicted from Eq. (2.15).

2.2 Real source caracterized by a gain

For a real source as an antenna, the emitted field is not isotropic but depends on the angles (θ, ϕ) defined in spherical coordinates. The function describing this phenomenon is the gain

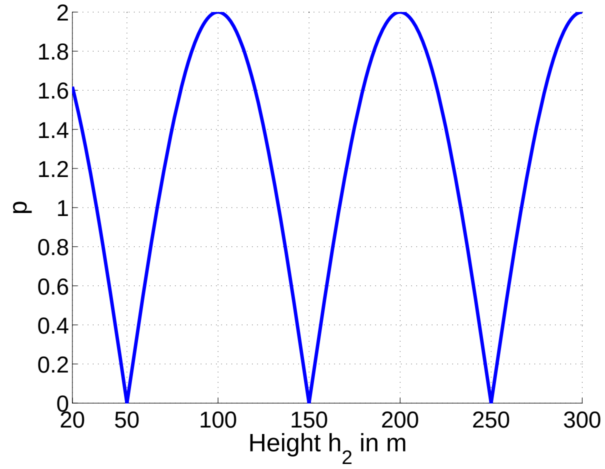

FIGURE 2.3 – p (Eq. (2.13)) versus the receiver height h_2 for $h_1 = 50$ m, $d = 10$ km and $f = 300$ MHz.

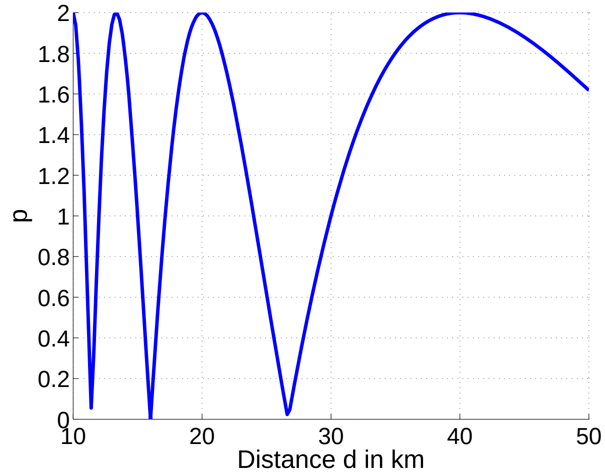

FIGURE 2.4 – p (Eq. (2.13)) versus the horizontal distance d for $h_1 = 100$ m, $h_2 = 200$ m and $f = 300$ MHz.

function $G(\theta, \phi) = \eta D(\theta, \phi)$, in which D is known as the directive gain and the number $0 < \eta < 1$ is related to the antenna efficiency. The directive gain signifies the ratio of radiated power in a given direction relative to that of an isotropic radiator which is radiating the same total power as the antenna in question but uniformly in all directions. Note that a true isotropic radiator does not exist in practice.

Now, we consider the problem shown in figure 2.5.

The power density p_E (in W/m^2) emitted by the antenna is

$$p_E = \frac{P_E G_E}{4\pi R^2} \tag{2.18}$$

where P_E is the emitted power (in W) and G_E the emitted antenna gain.

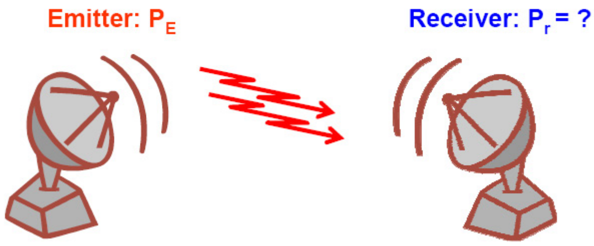

FIGURE 2.5 – Received power from an emitter.

The power (in W) received by the antenna is then

$$P_R = p_E A_R = \frac{P_E G_E A_R}{4\pi R^2} \tag{2.19}$$

where A_R is the effective aperture in m^2 of the received antenna. It is well known that A_R is related to the gain G_R of the received antenna by

$$A_R = \frac{G_R \lambda_0^2}{4\pi} \tag{2.20}$$

Then

$$P_R = \frac{P_E G_E}{4\pi R^2} \frac{G_R \lambda_0^2}{4\pi} = P_E G_E G_R \left(\frac{\lambda_0}{4\pi R}\right)^2 = \frac{P_E G_E G_R}{L_0} \tag{2.21}$$

where $L_0 > 1$ is called the **path loss** in free space. It is defined as

$$L_0 = \left(\frac{4\pi R}{\lambda_0}\right)^2 \tag{2.22}$$

In dB scale, $10\log_{10}(L_0)$, L_0 becomes

$$\begin{aligned} L_0 \text{ (dB)} = 10\log_{10}\left[\left(\frac{4\pi R}{\lambda_0}\right)^2\right] &= 20\log_{10}\left(\frac{4\pi}{c}\right) + 20\log_{10} R + 20\log_{10} f \\ &= 32.45 + 20\log_{10} R_{\text{km}} + 20\log_{10} f_{\text{MHz}} \end{aligned} \tag{2.23}$$

Thus, in dB scale, the path loss increases with the distance R and the frequency f.

2.3 Radar equation and Radar Cross section

In this subsection, the Radar cross section is introduced via the Radar equation.

We consider the problem shown in figure 2.6. An antenna illuminates an object. A part of the power reflected by the object returned toward the receiver. The purpose is to calculate the received power.

The emitted power density is

$$p_E = \frac{P_E G_E(\theta_E, \phi_E)}{4\pi R^2} \tag{2.24}$$

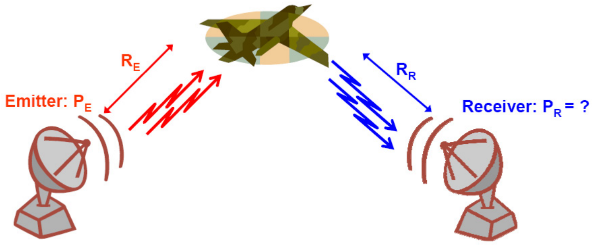

FIGURE 2.6 – Received power from an emitter illuminating an object.

where P_E is the emitted power (in W) and G_E the emitted antenna gain defined along the spherical angles (θ_E, ϕ_E).

From the distance R_E, the power (in W) reflected (diffracted) by this object is then

$$P_O = p_E|_{R=R_E}\, \sigma = \frac{P_E G_E(\theta_E, \phi_E)}{4\pi R_E^2}\sigma \tag{2.25}$$

where σ is the **Radar cross section** (in m^2) of the object. This magnitude caracterized the capacity of an object to reflect the power in the the specific directions (θ_R, ϕ_R) knowing the directions (θ_E, ϕ_E). In other words, it is a measure of how detectable an object is with a Radar. It is an **intrinsic** property of the object. It depends on

- The angles (θ_E, ϕ_E) and (θ_R, ϕ_R).
- The radar frequency f.
- The polarization of the incident electric field
- The shape of the object.
- The electric properties of the object (ϵ and μ).

The density power (in w/m^2) from the distance R_R is then

$$p_O = \frac{P_O}{4\pi R_R^2} \tag{2.26}$$

The received power (in W) is then

$$\begin{aligned} P_R &= p_O A_R = \frac{P_O}{4\pi R_R^2}\frac{G_R(\theta_R, \phi_R)\lambda_0^2}{4\pi} \\ &= \frac{P_E G_E(\theta_E, \phi_E) G_R(\theta_R, \phi_R)\lambda_0^2 \sigma}{(4\pi)^3 R_E^2 R_R^2} \end{aligned} \tag{2.27}$$

where $G_R(\theta_R, \phi_R)$ is the gain of the reception antenna in the directions (θ_R, ϕ_R).

The above equation is named as the **Radar equation** and it is then the basic equation used to calculate the power received by a Radar system.

When the receiver is the same as the emitter, corresponding to a **monostatic** configuration, we have $R_R = R_E = R$, $(\theta_R, \phi_R) = (\theta_E, \phi_E) = (\theta, \phi)$ and $G_R = G_E = G$. The above equation is then simplified as

$$P_R = \frac{P_E G^2(\theta, \phi)\lambda_0^2 \sigma}{(4\pi)^3 R^4} \tag{2.28}$$

2.4 Exercises

2.4.1 Exercise 1 : Reflexion by a ground

A radio link ($\lambda_0 = 2$ m) is established between a boat, for which its antenna is located at a height of $h_1 = 10$ m, and two receivers. The first one is located on the coast from a distance of $d = 10$ km and a height of $h_2 = 10$ m. The second one is located on the montain from a distance of $d = 12$ km and a height of $h_2 = 10$ m.

The reflection coefficient of the sea is assumed to be -1.

1. Show that de modulus of the received field is

$$|E| = 2\left|E_0 \sin\left(\frac{\phi}{2}\right)\right| \qquad \phi = \frac{4\pi h_1 h_2}{\lambda_0 d} \tag{E1}$$

   where E_0 is the incidend field.
2. Calculate $p = |E/E_0|$ for the two cases and give a physical interpretation.

2.4.2 Exercise 2 : Link satellite

The satellite Voyager 2 in 1993 was $R = 6^9$ km from the Earth. The power of its emitter was 20 W and its antenna gain was 48 dB. The used frequency is 8.4 GHz.

1. Calculate the power density p_E radiated on the Earth.
2. Calculate the power P_R transmitted to the receiver located on the Earth if the gain G_R of the parabolic antenna is 70 dB.
3. Calculate in dB the path loss L_0.
4. Calculate the diameter D of the antenna knowing that $G_R = (\pi D/\lambda_0)^2\eta$ with $\eta = 0.6$.

2.4.3 Exercise 3 : Radar Cross Section (RCS)

We consider a monostatic configuration (emitter and receiver are the same). The emitter illuminates an object of RCS σ.

1. Then show

$$\sigma = 4\pi R^2 \frac{|E_R|^2}{|E_E|^2} \tag{E2}$$

   where R is the distance from the receiver to the object, E_E is the emitted field and E_R the received field.
2. Why the rigorous definition of RCS is

$$\sigma = \lim_{R\to\infty} 4\pi R^2 \frac{|E_R|^2}{|E_E|^2} \tag{E3}$$