Energy conservation of the scattering from one-dimensional random rough surfaces in the high-frequency limit

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Energy conservation of the scattering from one-dimensional strongly rough dielectric surfaces is investigated using the Kirchhoff approximation with single reflection and by taking the shadowing phenomenon into account, both in reflection and transmission. In addition, because no shadowing function in transmission exists in the literature, this function is presented here in detail. The model is reduced to the high-frequency limit (or geometric optics). The energy conservation criterion is investigated versus the incidence angle, the permittivity of the lower medium, and the surface rms slope. © 2005 Optical Society of America

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The Kirchhoff theory in reflection (for instance, see Refs. 1–3) is well known, but in transmission few articles deal with this subject.[3–6] The purpose of this Letter is to study the Kirchhoff approximation in the high-frequency limit from a stationary rough dielectric surface by taking into account the shadowing phenomenon both in reflection and in transmission. The validity of this approximation is investigated through the energy conservation criterion. To our knowledge, no transmission shadowing function has been derived.

As depicted in Fig. 1, $E_i(\mathbf{r})=E_0 e^{i\mathbf{k}_i\cdot\mathbf{r}}$ (the time factor $e^{-i\omega t}$ is omitted) denotes the incident field of direction $\hat{\mathbf{k}}_i=(\hat{k}_i,\hat{q}_i)=(\sin\theta_i,-\cos\theta_i)$ with an incidence angle θ_i onto the surface S_0. In the same way, $\mathbf{E}_{r,t}$ denote the reflected and transmitted scattered fields of directions $\hat{\mathbf{k}}_{r,t}=(\hat{k}_{r,t},\hat{q}_{r,t})=(\sin\theta_{r,t},\pm\cos\theta_{r,t})$ with scattering angles $\theta_{r,t}$. A point out of the surface S_0 is denoted as $\mathbf{r}=(x,z)$, and a point on S_0 is denoted as $\mathbf{r}_0=[x_0,\zeta(x_0)]$. The local surface slope is $\gamma(x_0)=\zeta'(x_0)$, and its local normal unit vector is $\hat{\mathbf{n}}_0$.

The model starts from the Kirchhoff–Helmholtz equations in reflection and transmission, respectively[6]:

$$E_r(\mathbf{r}) = +\int\left[E(\mathbf{r}_0)\frac{\partial G_1}{\partial n_0}-\frac{\partial E(\mathbf{r}_0)}{\partial n_0}G_1\right]\mathrm{d}S_0,$$

$$E_t(\mathbf{r}) = -\int\left[E_t(\mathbf{r}_0)\frac{\partial G_2}{\partial n_0}-\frac{\partial E_t(\mathbf{r}_0)}{\partial n_0}G_2\right]\mathrm{d}S_0. \quad (1)$$

$G_{1,2}\equiv G_{1,2}(\mathbf{r}_0,\mathbf{r})=(i/4)H_0^{(1)}(k_{1,2}\|\mathbf{r}_0-\mathbf{r}\|)$ is the two-dimensional Green's function, where $k_{1,2}=k_0\sqrt{\epsilon_{r1,r2}}$ (k_0 is the wave number in the vacuum). In the far-field zone, it can be approximated as follows:

$$G_{1,2}(\mathbf{r}_0,\mathbf{r})\simeq\frac{i}{4}\sqrt{\frac{2}{\pi k_{1,2}r}}\exp(-i\pi/4)$$
$$\times\exp[i(k_{1,2}r-\mathbf{k}_{r,t}\cdot\mathbf{r}_0)]. \quad (2)$$

By applying the Kirchhoff approximation both in reflection and transmission, we get the expressions of the total and transmitted fields at $\mathbf{r}_0$, E, and E_t, for the reflection and the transmission cases, respectively:

$$E(\mathbf{r}_0)=[1+\mathcal{R}(\theta)]E_i(\mathbf{r}_0),$$

$$\frac{\partial E(\mathbf{r}_0)}{\partial n_0},=i(\mathbf{k}_i\cdot\hat{\mathbf{n}}_0)[1-\mathcal{R}(\theta)]E_i(\mathbf{r}_0), \quad (3)$$

$$E_t(\mathbf{r}_0)=\mathcal{T}(\theta)E_i(\mathbf{r}_0),$$

$$\frac{\partial E_t(\mathbf{r}_0)}{\partial n_0}=i(\mathbf{k}_{tsp}\cdot\hat{\mathbf{n}}_0)\mathcal{T}(\theta)E_i(\mathbf{r}_0), \quad (4)$$

where θ is the local incidence angle from the local normal to the surface $\hat{\mathbf{n}}_0$. $\mathcal{R}$ and $\mathcal{T}$ are the Fresnel reflection and transmission coefficients, respectively. $\mathbf{k}_{tsp}$ is directed according to the specular direction of transmission.

Substituting Eqs. (3) and (4) and expression (2) into Eqs. (1), with $\mathrm{d}S_0=[1+\gamma^2(x_0)]^{1/2}\mathrm{d}x_0$, the reflected and transmitted fields can be simplified. Then, the stationary phase approximation is used. It assumes

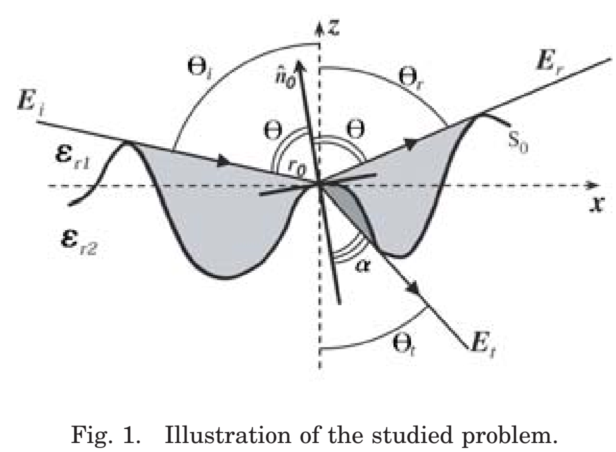

Fig. 1. Illustration of the studied problem.

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that the main contribution of the field scattered by the surface comes from regions around the local specular direction of the facet. That is to say, the phase term $g(x,z)$ in the exponential is such that $\partial g(x,z)/\partial z=0$. Thus, only the surface slopes $\gamma_{r,t}^0$ that specularly reflect or transmit the field in the specular direction are considered. These slopes are given by

$$\gamma_r^0=-\frac{\hat{k}_r-\hat{k}_i}{\hat{q}_r-\hat{q}_i},\quad \gamma_t^0=-\frac{k_2\hat{k}_t-k_1\hat{k}_i}{k_2\hat{q}_t-k_1\hat{q}_i}. \tag{5}$$

With this approximation, the local incidence angle θ for the reflected field can be expressed as $\theta=|\theta_i+\theta_r|/2$, and the local incidence angle $\theta\equiv\chi$ for the transmitted field can be expressed as $\cos(\chi)=\mathrm{sgn}(k_2\hat{q}_t-k_1\hat{q}_i)\{[k_1-k_2(\hat{k}_t\hat{k}_i+\hat{q}_t\hat{q}_i)]/[k_1{}^2+k_2{}^2-2k_1k_2\times(\hat{k}_t\hat{k}_i+\hat{q}_t\hat{q}_i)]^{1/2}\}$. In Eq. (1), the dependence over slope γ can be suppressed, and one gets

$$\frac{E_r^\infty(\mathbf{r})}{E_0}=-\frac{\exp[i(k_1r+\pi/4)]}{\sqrt{2\pi k_1 r}}\mathcal{R}(\theta)f_r(\theta_i,\theta_r)\times\int_{-L_0}^{+L_0}\Xi(x_0)\exp[i(\mathbf{k}_i-\mathbf{k}_r)\cdot\mathbf{r}_0]\mathrm{d}x_0, \tag{6}$$

$$\frac{E_t^\infty(\mathbf{r})}{E_0}=-\frac{\exp[i(k_2r-\pi/4)]}{\sqrt{2\pi k_2 r}}\mathcal{T}(\chi)f_t(\theta_i,\theta_t)\times\int_{-L_0}^{+L_0}\Xi(x_0)\exp[i(\mathbf{k}_i-\mathbf{k}_t)\cdot\mathbf{r}_0]\mathrm{d}x_0, \tag{7}$$

where $f_r(\theta_i,\theta_r)=(1-\hat{k}_r\hat{k}_i-\hat{q}_r\hat{q}_i)/(\hat{q}_r-\hat{q}_i)$, $f_t(\theta_i,\theta_t)=[k_2-k_1(\hat{k}_t\hat{k}_i+\hat{q}_t\hat{q}_i)]/(k_2\hat{q}_t-k_1\hat{q}_i)$, $2L_0$ is the illuminated surface length, and $\Xi(x_0)$ the illumination function, which is equal to 1 if the point $\mathbf{r}_0$ is seen by both the emitter and the receiver, and 0 otherwise.

With the latter expressions, one can express the incoherent scattering coefficients in reflection and transmission $\sigma_{r,t}$, defined by $\sigma_{r,t}(\theta_i,\theta_{r,t})=Rp_{r,t}(\theta_i,\theta_{r,t})/|E_i|^2 2L_0\cos\theta_i$, with $p_{r,t}=\langle E_{r,t}^\infty E_{r,t}^{\infty'*}\rangle-|\langle E_{r,t}^\infty\rangle|^2$. $E_{r,t}^{\infty'}$ is the reflected or transmitted far field, in which the surface integration is taken over the point $\mathbf{r}_0'$ distinct from the point $\mathbf{r}_0$.

The geometric optics approximation or high-frequency limit ($k\sigma_h>1$, with σ_h being the surface rms height) assumes that the scattering intensity contributes for only closely located correlated points on the surface $(\mathbf{r}_0,\mathbf{r}_0')$ compared to the surface correlation length. With this approximation, the coherent contribution $|\langle E_{r,t}^\infty\rangle|^2$ can be neglected. Moreover, the phase term inside the exponential can be expanded as $(\mathbf{k}_i-\mathbf{k}_r)\cdot(\mathbf{r}_0-\mathbf{r}_0')=k_1[(\hat{k}_r-\hat{k}_i)+(\hat{q}_r-\hat{q}_i)\gamma(x_0)](x_0'-x_0)$ and $(\mathbf{k}_i-\mathbf{k}_t)\cdot(\mathbf{r}_0-\mathbf{r}_0')=[(k_2\hat{k}_t-k_1\hat{k}_i)+(k_2\hat{q}_t-k_1\hat{q}_i)\gamma(x_0)]\times(x_0'-x_0)$. Thus, as demonstrated by Bourlier and Berginc,[7] one can eventually obtain

$$\sigma_r=\frac{|\mathcal{R}|^2(\theta)}{\cos\theta_i}f_r^2(\theta_i,\theta_r)\frac{p_s(\gamma_r^0)}{|\hat{q}_r-\hat{q}_i|}S_{11}(\theta_i,\theta_r|\gamma_r^0),$$

$$\sigma_t=\frac{|\mathcal{T}|^2(\chi)}{\cos\theta_i}f_t^2(\theta_i,\theta_t)\frac{p_s(\gamma_t^0)}{\left|\hat{q}_t-\dfrac{k_1}{k_2}\hat{q}_i\right|}S_{12}(\theta_i,\theta_t|\gamma_t^0), \tag{8}$$

where $S_{11}(\theta_i,\theta_r|\gamma_r^0)$ and $S_{12}(\theta_i,\theta_t|\gamma_t^0)$ are the bistatic average shadowing functions in reflection and transmission, respectively.

The expression of the reflection and transmission bistatic statistical shadowing functions $S_{11}(\theta_i,\theta_r|\zeta_0,\gamma_0)$ and $S_{12}(\theta_i,\theta_t|\zeta_0,\gamma_0)$ are obtained from the monostatic statistical shadowing functions $S_1(\theta_1|\zeta_0,\gamma_0)$ and $S_2(\theta_2|\zeta_0,\gamma_0)$ ($\theta_{1,2}$ are any angles inside media 1 and 2, respectively). The expression of the shadowing function inside upper medium 1, $S_1(\theta_1|\zeta_0,\gamma_0)$, was given by Bourlier *et al.*[8] This function represents the probability that the ray of angle θ_1 does not intercept the surface before striking it at the point $\mathbf{r}_0=[x_0,\zeta_0\equiv\zeta(x_0)]$, with $\gamma_0=\zeta_0'$:

$$S_1(\theta_1|\zeta_0,\gamma_0)=\Upsilon(\mu_1-\gamma_0)[P_h(\zeta_0)-P_h(-\infty)]^{\Lambda(\mu_1)}, \tag{9}$$

where P_h is a primitive of the height probability density function (pdf) p_h; $\Upsilon(x)=1$ if $x\geqslant 0$, and 0 otherwise (Υ is the unit step function); and Λ is defined as $\Lambda(\mu_1)=(1/\mu_1)\int_{\mu_1}^{+\infty}(\gamma-\mu_1)p_s(\gamma)\,\mathrm{d}\gamma$, with $\mu_1=|\cot(\theta_1)|$, where p_s denotes the slope pdf.

In this Letter the shadowing function inside the lower medium S_2 is derived from the expression of S_1. The only difference is that the surface S_0 is illuminated from underneath, so that the points of the surface that are concerned by the shadow are no longer $\zeta\in]-\infty;\zeta_0]$ as in Eq. (9), but $\zeta\in[\zeta_0;+\infty[$. Then, the expression of S_2 is

$$S_2(\theta_2|\zeta_0,\gamma_0)=\Upsilon(\mu_2-\gamma_0)\{1-[P_h(\zeta_0)-P_h(-\infty)]\}^{\Lambda(\mu_2)}, \tag{10}$$

where $\mu_2=|\cot\theta_2|$ is the orientation of the transmitted beam inside lower medium 2 with a transmission angle θ_2.

Then, the bistatic statistical shadowing functions S_{11} and S_{12} are expressed from S_1 and S_2. The expression of S_{11} was given by Bourlier *et al.*[8]: $S_{11}(\theta_i,\theta_r|\zeta_0,\gamma_0)$ equals $S_1(\theta_r|\zeta_0,\gamma_0)$ if $\theta_r\in[-\pi/2;-|\theta_i|[$, $S_1(\theta_i|\zeta_0,\gamma_0)$ if $\theta_r\in[-|\theta_i|;0[$, and $S_1(\theta_i|\zeta_0,\gamma_0)S_1(\theta_r|\zeta_0,\gamma_0)$ if $\theta_r\in[0;+\pi/2]$. Using the same method, we show that $S_{12}(\theta_i,\theta_t|\zeta_0,\gamma_0)=S_1(\theta_i|\zeta_0,\gamma_0)S_2(\theta_t|\zeta_0,\gamma_0)\,\forall(\theta_i,\theta_t)$.

In the stationary phase approximation, the surface slope $\gamma_0\equiv\gamma_{r,t}^0$ is given by Eq. (5). The statistical averaging over ζ_0 and γ_0 gives the average shadowing functions $S_{11}(\theta_i,\theta_r|\gamma_r^0)$ and $S_{12}(\theta_i,\theta_t|\gamma_t^0)$, expressed for any random process by

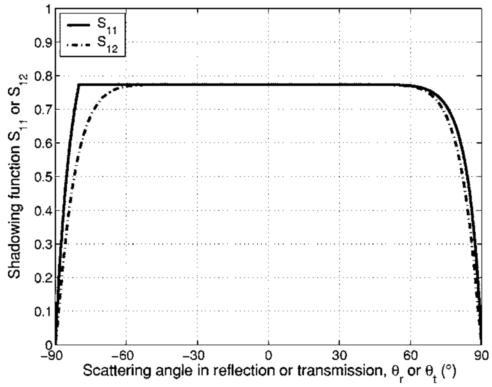

Fig. 2. Comparison between S_{11} and S_{12}, with $\theta_i=80°$ and slope rms $\sigma_s=0.3$.

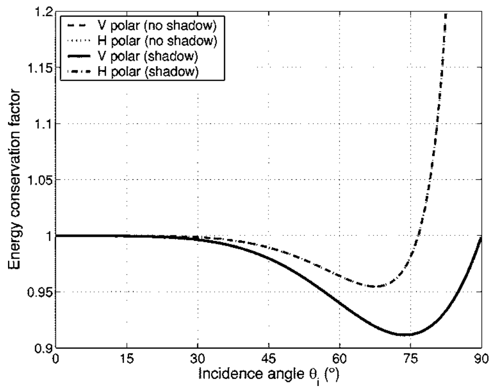

Fig. 3. Simulations for $\sigma_s=0.2$ and $\epsilon_{r2}=i\infty$ (case of a perfectly conducting surface: V polar≡H polar).

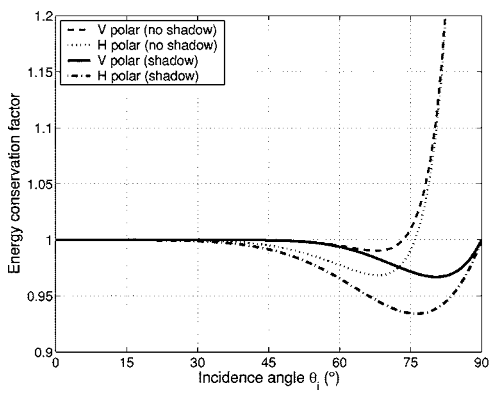

Fig. 4. Simulations for $\sigma_s=0.2$ and $\epsilon_{r2}=4$.

$$S_{11}(\theta_i,\theta_r|\gamma_r^0)=\begin{cases}[1+\Lambda(\mu_r)]^{-1}(\theta_r\in[-\frac{\pi}{2};-|\theta_i|[)\\ [1+\Lambda(\mu_i)]^{-1}(\theta_r\in[-|\theta_i|;0[),\\ [1+\Lambda(\mu_i)+\Lambda(\mu_r)]^{-1}(\theta_r\in[0;\frac{\pi}{2}])\end{cases}\tag{11}$$

$$S_{12}(\theta_i,\theta_t|\gamma_t^0)=B[1+\Lambda(\mu_i),1+\Lambda(\mu_t)],\tag{12}$$

where B is the beta function, also called the Eulerian integral of the first kind.[9] A comparison between the reflection and transmission average shadowing functions, S_{11} and S_{12}, respectively, is illustrated in Fig. 2 for Gaussian statistics, where $\Lambda(v)=e^{-v^2}/(2\sqrt{\pi})-\mathrm{erfc}(v)/2$, with $v=\mu/(\sqrt{2}\sigma_s)$.

The interest of this Letter is to study the energy conservation of this model. Thus, one defines

$$\frac{P_{r,t}}{P_i}=\int_{-\pi/2}^{+\pi/2}\sigma_{r,t}(\theta_i,\theta_{r,t})\mathrm{d}\theta_{r,t}.\tag{13}$$

The study of energy conservation then consists of evaluating the quantity $\eta=\{P_r+[(\epsilon_{r2}/\epsilon_{r1})]^{1/2}P_t\}/P_i$ in comparison with 1 (by analogy with the case of a plane interface, where $|\mathcal{R}|^2+[(\epsilon_{r2}/\epsilon_{r1})]^{1/2}(\cos\theta_t/\cos\theta_i)|\mathcal{T}|^2=1$). For a perfectly conducting surface, we can obtain an analytical closed form of $\eta(\theta_i,\sigma_s)\,\forall\,\theta_i$. In particular, for $\theta_i=0°$, we can show that $\eta=\mathrm{erf}[1/(\sigma_s\sqrt{2})]$, such that if $\eta>0.99$, $\sigma_s<0.388$, and if $\eta>0.999$, $\sigma_s<0.304$. Thus, the energy conservation factor η is all the better as the slope rms is low, since the multiple reflections occur for high slope rms. The comparison has been done with various slope rms σ_s and between the model without shadow and the model with shadow. The results are presented in Figs. 3 and 4, for vertical (V) and horizontal (H) polarizations.

In Figs. 3 and 4, one can see a good energy conservation rate for low angles, for both polarizations, and without and with shadow. However, the model without shadow diverges for grazing angles. On the contrary, the model with shadow tends to 1 when θ_i tends to 90°. Moreover, for grazing angles, the model with shadow decreases when θ_i increases to reach a minimum, and then increases to tend to 1 at 90°. This is due to the multiple scattering effect, which occurs for grazing angles. As θ_i increases, the multiple scattering effect increases; still, after a given angle its influence begins to decrease because of the shadow and tends to 0 when θ_i tends to 90°.

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