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Theoretical study of the Kirchhoff integral from a two-dimensional randomly rough surface with shadowing effect: application to the backscattering coefficient for a perfectly-conducting surface

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Abstract
In this paper, the backscattering coefficient of a two-dimensional randomly rough perfectly-conducting surface is investigated using the Kirchhoff approach with a shadowing function. The rough surface height/slope correlations assumed to be Gaussian are accounted for in this analysis. The scattering coefficient is then formulated in terms of a characteristic function for the integrations over the surface heights, in terms of expected values for the integrations over the surface slopes. Numerical comparisons of Kirchhoff's approach (KA) with the stationary-phase (SP) approximation are made with respect to the choice of the one-dimensional surface height autocorrelation function and the shadowing effect. For an isotropic surface the results show that SP underestimated the incoherent backscattering coefficient compared with KA. Moreover, when the correlation between the slopes and the heights is neglected, the shadowing effect may be ignored.

1. Introduction

The problem of electromagnetic scattering from natural surfaces is a matter of great relevance from both a theoretical and an application point of view. This problem is of interest in many research areas, including remote sensing of the environment, medical imaging, sonar, optics and astronomy [1]. The scattering of electromagnetic waves by rough surfaces has been studied for many years, but no exact closed-form solution has been obtained. Numerical techniques such as the method of moments can be used to compute the exact solution, but, in general, these techniques are computationally prohibitive. We can also quote the integral equation based on a Monte Carlo technique (Thorsos *et al* [2,3]). Usually, when dealing with practical applications, approximate models are examined. Among the many surface-scattering theories, the small-slope approximation (SSA) developed by Voronovich [4] and the perturbation approximation can be used (Thorsos and Broschat [5–7] and Berginc and Chevalier [8]). The Kirchhoff approximation or physical optics, which is the most widely used (Olgivy [1], Ulaby and co-workers [9, 10] and Beckmann and Spizzichino [11]) is investigated in this paper. This approach is valid if the radius of curvature at every point on the surface is large relative to the electromagnetic wavelength λ, and if the correlation length L_c is larger than λ. The Kirchhoff approximation is used as a starting point for high-frequency analysis when the geometric optics approximation is obtained by applying the stationary phase method. We can notice that [1] gives a computable expression for the Kirchhoff integral limited to the class of known surfaces. To evaluate the Kirchhoff integral involving a nonlinear function of the correlation function is a difficult exercise.

With Kirchhoff's approximation, the scattered field from a rough surface is expressed as an integral over the surface [9] with the integrand depending on five variables: the surface (two variables), the slopes (two variables) and the height surface. This means that the determination of the scattering coefficient obtained from averaging the scattered field multiplied by its conjugate requires ten integrations. Assuming a stationary process of the surface slope and height joint probability density (the surface height autocorrelation function depends only on the distance between two points on the surface), the ten integrations are reduced to eight integrations. However, as it stands, the scattering coefficient integrand remains a complicated function of the slopes, whereas the heights appear in the exponential phase. For a monostatic configuration (transmitter and receiver located at the same place corresponding to the backscattering), the dependence of the scattered field with respect to the surface slopes can be expressed analytically as functions of the Fresnel coefficients, which also depend on the slopes. For a perfectly-conducting surface the Fresnel coefficients in vertical and horizontal polarizations are equal to 1 and -1, respectively. Therefore, the integration over the surface slopes of the backscattering coefficient can be performed analytically with a surface Gaussian joint height and slope probability density.

In this paper, by applying the Kirchhoff approximation, the backscattering coefficient by a perfectly-conducting stationary surface with shadowing effect is computed and compared with the stationary phase solution. The average of the backscattering coefficient uses a Gaussian surface slope and height joint probability density function (PDF) defined by Bourlier *et al* [12] where the multiple scattering is neglected. Moreover, in the average of the scattering coefficient, the two-dimensional shadowing effect is introduced. This means that the PDF with shadow has to be determined. Sancer [13] studied the shadowing effect on the scattering coefficient obtained from the Kirchhoff theory. He showed that under the geometrical optics approximation, the shadowing function is statistically independent of the unshadowed scattering coefficient. Strictly speaking, it is exact only if the shadowing function is assumed to be independent of height and slope surface. From Smith's approach [14, 15], Bourlier *et al* [12, 16] studied the statistical monostatic one- and two-dimensional shadowing functions with correlation and they showed that they depend on the height and slope surface. For an uncorrelated process, Smith's monostatic two-dimensional statistical shadowing function depends only on the heights with a restriction over the slopes. Since the difference between the correlated and uncorrelated Smith shadowing functions is very small, the uncorrelated statistical shadowing function is used.

The plan of this work is as follows. The scattered field is determined from Kirchhoff theory in section 2. The backscattering coefficient is calculated in section 3 for any surface height autocorrelation function. In section 4, assuming a Gaussian PDF with correlation [12] and including the shadowing effect, the previous results are applied to a perfectly-conducting surface. In the final section, the incoherent backscattering coefficient is simulated for an isotropic surface, and compared with the stationary phase method.

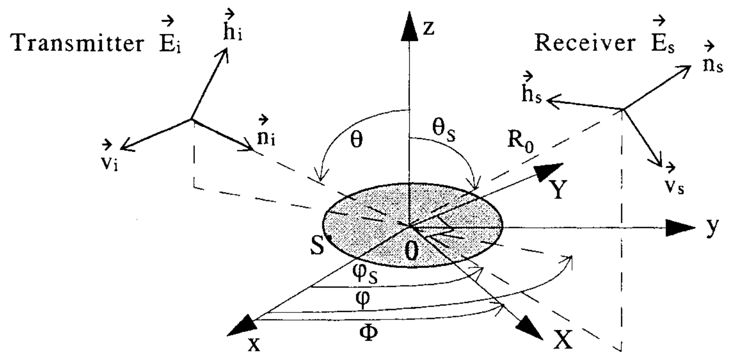

Figure 1. Geometry of the problem.

2. Scattered field with Kirchhoff approach

With the Kirchhoff approach, the scattered field is written in terms of the tangential field on a rough surface. The surface field is then approximated by the field that would be present if the rough surface were replaced by a planar surface tangential to the point of interest. With this assumption, the scattered field is expressed as [9]

$$\vec{E}^s = K\vec{n}_s \wedge \iint [\vec{n} \wedge \vec{E} - \eta\vec{n}_s \wedge (\vec{n} \wedge \vec{H})] \exp[jk(\vec{n}_s - \vec{n}_i) \cdot \vec{r}'] \, dS' \tag{1}$$

where $K = -jk\exp(-jkR_0)/(4\pi R_0)$, with k being the wavenumber in the medium where the field is evaluated, R_0 ranges from the centre of the illuminated area S' to the point of observation (see figure 1), $\vec{n}_s, \vec{n}_i$ are the unit vectors in the scattered and incident directions, respectively, defined in spherical coordinates as

$$\begin{aligned}
\vec{n}_i &= \sin\theta\cos\varphi\vec{x} + \sin\theta\sin\varphi\vec{y} - \cos\theta\vec{z} \\
\vec{n}_s &= \sin\theta_s\cos\varphi_s\vec{x} + \sin\theta_s\sin\varphi_s\vec{y} + \cos\theta_s\vec{z}
\end{aligned} \tag{2}$$

where $\vec{x}, \vec{y}, \vec{z}$ are unit vectors in Cartesian coordinates. $\vec{r}' = x'\vec{x} + y'\vec{y} + z'\vec{z}$ is the vector indicating the location of the surface point according to the centre of the illuminated area. In (1), the total $\vec{n} \wedge \vec{E}$ electric and magnetic $\eta\vec{n} \wedge (\vec{n} \wedge \vec{H})$ tangential fields are given by [9]

$$\begin{aligned}
\vec{n} \wedge \vec{E} &= [(1 + R_H)(\vec{a} \cdot \vec{t})(\vec{n} \wedge \vec{t}) - (1 - R_V)(\vec{n} \cdot \vec{n}_i)(\vec{a} \cdot \vec{d})\vec{t}]E_0 \\
\eta(\vec{n} \wedge \vec{H}) &= -[(1 + R_V)(\vec{a} \cdot \vec{d})(\vec{n} \wedge \vec{t}) + (1 - R_H)(\vec{n} \cdot \vec{n}_i)(\vec{a} \cdot \vec{t})\vec{t}]E_0
\end{aligned} \tag{3}$$

where E_0 is the magnitude of the incident field with a unit polarization vector $\vec{a}$. $\{R_V, R_H\}$ denote the Fresnel coefficients in the vertical V and horizontal H polarizations determined with an incidence angle $\theta_1 = \arccos[\vec{n} \cdot \vec{n}_i]$, where $\vec{n}$ is the unit vector normal to the local surface defined as $\vec{n} = (-\gamma_x\vec{x} - \gamma_y\vec{y} + \vec{z})/\sqrt{1 + \gamma_x^2 + \gamma_y^2}$, where the slopes $\{\gamma_x, \gamma_y\}$ are expressed as $\{\gamma_x = \partial z'/\partial x', \gamma_y = \partial z'/\partial y'\}$. $\{\vec{t}, \vec{d}\}$ are given by $\vec{t} = \vec{n}_i \wedge \vec{n}/\|\vec{n}_i \wedge \vec{n}\|$ and $\vec{d} = \vec{n}_i \wedge \vec{t}$.

In the forward scattering alignment (FSA) convention [10] the incident and scattered unit polarization vectors denote $\{\vec{h}_i, \vec{v}_i, \vec{h}_s, \vec{v}_s\}$ defined as

$$\begin{aligned}
\vec{h}_i &= -\sin\varphi\vec{x} + \cos\varphi\vec{y} \\
\vec{v}_i &= \vec{h}_i \wedge \vec{n}_i = -\cos\theta\cos\varphi\vec{x} - \cos\theta\sin\varphi\vec{y} - \sin\theta\vec{z}
\end{aligned} \tag{4}$$

and

$$\vec{h}_s = -\sin\varphi_s \vec{x} + \cos\varphi_s \vec{y}$$
$$\vec{v}_s = \vec{h}_s \wedge \vec{n}_s = \cos\theta_s \cos\varphi_s \vec{x} + \cos\theta_s \sin\varphi_s \vec{y} - \sin\theta_s \vec{z} \tag{5}$$

The integrand of (1), which is a complicated function of the slopes $\{\gamma_x, \gamma_y\}$, then depends on two deterministic variables $\{x', y'\}$ and three random variables $\{z', \gamma_x, \gamma_y\}$. This expression can be simplified if the backscattered field is studied, which involves that $\{\vec{n}_i = -\vec{n}_s, \vec{v}_i = \vec{v}_s, \vec{h}_i = -\vec{h}_s\}$ and (1) becomes, with $\mathrm{d}S' = \mathrm{d}x'\,\mathrm{d}y'/(\vec{n}\cdot\vec{z})$

$$\vec{E}^S_{pq} = 2KE_0 \iint F_{pq} \exp\{\mathrm{j}[q_x x' + q_y y' + q_z z''(x', y')]\}\,\mathrm{d}x'\,\mathrm{d}y' \tag{6}$$

with the polarization term F_{pq} equal to $(p = \{\vec{h}_i, \vec{v}_i\}, q = \{\vec{h}_s, \vec{v}_s\})$

$$\begin{aligned} F_{h_s h_i} &= -F_{h_s h_s} = D_0(\vec{n}'\cdot\vec{n}_s)[R_H(\vec{n}'\cdot\vec{v}_s)^2 - R_V(\vec{n}'\cdot\vec{h}_s)^2] \\ F_{v_s v_i} &= F_{v_s v_s} = D_0(\vec{n}'\cdot\vec{n}_s)[R_H(\vec{n}'\cdot\vec{h}_s)^2 - R_V(\vec{n}'\cdot\vec{v}_s)^2] \\ F_{v_s h_i} &= F_{h_s v_i} = F_{v_s h_s} = D_0(\vec{n}'\cdot\vec{n}_s)(\vec{n}'\cdot\vec{v}_s)(\vec{n}'\cdot\vec{h}_s)(R_H + R_V) \end{aligned} \tag{6a}$$

with $\vec{n}' = (-\gamma_x\vec{x} - \gamma_y\vec{y} + \vec{z})$, $D_0 = 1/\|\vec{n}_s \wedge \vec{n}'\|^2$, $q_x = 2k\sin\theta_s\cos\varphi_s$, $q_y = 2k\sin\theta_s\sin\varphi_s$ and $q_z = 2k\cos\theta_s$. Note that $F_{v_s h_s} = F_{h_s v_s}$, as is demanded by reciprocity for backscattering. From (5), the scalar products of (6*a*) are expressed as

$$\begin{aligned} \vec{n}'\cdot\vec{n}_s &= \sin\theta_s(\gamma_x\cos\varphi_s + \gamma_y\sin\varphi_s) - \cos\theta_s \\ \vec{n}'\cdot\vec{v}_s &= -\cos\theta_s(\gamma_x\cos\varphi_s + \gamma_y\sin\varphi_s) - \sin\theta_s \\ \vec{n}'\cdot\vec{h}_s &= -\gamma_y\cos\varphi_s + \gamma_x\sin\varphi_s \end{aligned} \tag{6b}$$

and

$$\begin{aligned} D_0^{-1} = {} & \sin^2\theta_s[1 + (\gamma_y\cos\varphi_s - \gamma_x\sin\varphi_s)^2] + \cos^2\theta_s(\gamma_x^2 + \gamma_y^2) \\ & + \sin(2\theta_s)(\gamma_x\cos\varphi_s + \gamma_y\sin\varphi_s). \end{aligned} \tag{6c}$$

Thus, we can see for a monostatic configuration, that the polarization term F_{pq} remains a complicated function of the surface slopes.

For a perfectly-conducting surface, we have $\{R_V = 1, R_H = -1\}$, and (6*a*) becomes

$$F_{h_s h_s} = F_{v_s v_s} = -\vec{n}'\cdot\vec{n}_s \qquad F_{v_s h_s} = 0. \tag{7}$$

Since the surface is assumed to be perfectly conducting, the cross-polarization is equal to zero, whereas both polarizations are equal.

3. Backscattering coefficient

The scattering coefficient σ_{pq} for an extended target can be written as [9]

$$\sigma_{pq} = \frac{4\pi R_0^2\langle E^S_{pq} E^{S\prime *}_{pq}\rangle}{A_0|E_0^2|} \tag{8}$$

where A_0 is the illuminated area, and the symbol $\langle\cdots\rangle$ is the ensemble average. Substituting (6) into (8), the scattering coefficient is given by

$$\begin{aligned} \sigma_{pq} = \frac{k^2}{\pi A_0} \iiiint & \langle F_{pq}(\gamma_y, \gamma_y)\, F'^{*}_{pq}(\gamma'_x, \gamma'_y)\, \exp[\mathrm{j}q_z(z' - z'')]\rangle \\ & \times \exp\{\mathrm{j}[q_x(x' - x'') + q_y(y' - y'')]\}\,\mathrm{d}x'\,\mathrm{d}y'\,\mathrm{d}x''\,\mathrm{d}y'' \end{aligned} \tag{9}$$

where the symbol $*$ denotes a complex conjugate. Since the surface is assumed to be stationary, the surface spatial autocorrelation function depends only on the spatial differences of variables $\{u = x' - x'', v = y' - y''\}$. Moreover, assuming either that the illuminated surface size is infinite or much larger than the correlation length, the variables transformation $\{u = r\cos\Phi, v = r\sin\Phi\}$ in (9) leads to the following equation from [9, pp 934–5]:

$$\sigma_{pq} = \frac{k^2}{\pi}\int_0^{\infty} r\,\mathrm{d}r \int_0^{2\pi} \mathrm{d}\Phi\,\langle\cdots\rangle \exp[\mathrm{j}r(q_x^2+q_y^2)^{1/2}\cos(\Phi-\varphi_s)] \tag{10}$$

with

$$\langle\cdots\rangle = \iiiint\!\!\iint F_{pq}(\gamma_x,\gamma_y)\,F_{pq}'^{*}(\gamma_x',\gamma_y')\exp[\mathrm{j}q_z(z'-z'')]\,p(\vec{V}_{xy})\,\mathrm{d}\vec{V}_{xy}. \tag{10a}$$

Since $F_{pq}F_{pq}'^{*}\exp[\mathrm{j}q_z(z'-z'')]$ depends on the vector $\vec{V}_{xy}^{T} = [z'z''\gamma_x\gamma_x'\gamma_y\gamma_y']$ the average requires six integrations. $p(\vec{V}_{xy})$ denotes the surface height and slope joint probability density assumed to be Gaussian and expressed in Cartesian coordinates.

In polar coordinates, $p(\vec{V}_{xy})$ becomes $p(\vec{V}_{XY})$ [12, p 272]

$$p(\vec{V}_{XY}) = \frac{1}{(2\pi)^3|[C_{XY}]|^{1/2}}\exp\left(-\tfrac{1}{2}\vec{V}_{XY}^{T}[C_{XY}]^{-1}\vec{V}_{XY}\right) \tag{11}$$

where $|[C_{XY}]|$ is the determinant of the covariance matrix $[C_{XY}]$ expressed in the $\{(0X), (0Y), (0z)\}$ base in polar coordinates $\{r, \Phi\}$ (see figure 1) as

$$[C_{XY}] = \begin{bmatrix} \omega^2 & R_0 & 0 & R_1 & 0 & C_{16} \\ R_0 & \omega^2 & -R_1 & 0 & -C_{16} & 0 \\ 0 & -R_1 & \sigma_X^2 & -R_2 & \sigma_{XY}^2 & -C_{36} \\ R_1 & 0 & -R_2 & \sigma_X^2 & -C_{36} & \sigma_{XY}^2 \\ 0 & -C_{16} & \sigma_{XY}^2 & -C_{36} & \sigma_Y^2 & -C_{56} \\ C_{16} & 0 & -C_{36} & \sigma_{XY}^2 & -C_{56} & \sigma_Y^2 \end{bmatrix} \tag{12}$$

with

$$\begin{aligned} R_0 &= R_{00} - \cos(2\Phi)R_{02} \\ R_1 &= R_{10} - \cos(2\Phi)R_{12} \qquad R_{ij} = \frac{\mathrm{d}^i R_{0j}}{\mathrm{d}r^i} \\ R_2 &= R_{20} - \cos(2\Phi)R_{22} \end{aligned} \tag{12a}$$

$$\begin{aligned} \sigma_X^2 &= \alpha + \beta\cos(2\Phi) \\ \sigma_Y^2 &= \alpha - \beta\cos(2\Phi) \qquad \alpha = \frac{\sigma_x^2+\sigma_y^2}{2} \qquad \beta = \frac{\sigma_x^2-\sigma_y^2}{2} \\ \sigma_{XY}^2 &= -\beta\sin 2\Phi \end{aligned} \tag{12b}$$

$$\begin{aligned} C_{16} &= \frac{2R_{02}\sin(2\Phi)}{r} \\ C_{36} &= \frac{2\sin(2\Phi)}{r^2}(rR_{12} - R_{02}) \\ C_{56} &= \frac{R_{10}}{r} + \frac{\cos(2\Phi)}{r^2}(4R_{02} - rR_{12}). \end{aligned} \tag{12c}$$

$R_0(r, \Phi)$ is the surface height two-dimensional autocorrelation function in polar coordinates, whereas $-R_2$ is the surface slope two-dimensional autocorrelation function. $\{R_{00}(r), R_{02}(r)\}$ represent the isotropic and anisotropic parts of R_0 and Φ the azimuthal direction according to the $(0x)$-direction, which characterizes the anisotropic effect. For example, when a sea surface is considered, this term corresponds to the wind direction according to the $(0x)$-axis. $\{\sigma_X^2 = -R_2(0,\Phi), \sigma_Y^2 = -C_{56}(0,\Phi)\}$ denotes the surface slope variances in the $\{(0X), (0Y)\}$ directions, respectively, and $\sigma_{XY}^2 = -C_{36}(0,\Phi)$ the surface slope cross-variance. $\omega^2 = R_0(0,\Phi) = I_0(0)$ is the surface height variance with $R_{02}(0) = 0$.

From (6*a*) and (10), we can see that the computation of the backscattering coefficient requires height numerical integrations over $\{r, \Phi, z', z'', \gamma_x, \gamma_x', \gamma_y, \gamma_y'\}$, since the analytical integrations are impossible. Moreover, the determination of the probability density requires the inversion of the covariance matrix $[C_{XY}]$.

The introduction of the shadowing function modifies the surface height $\{z', z''\}$ probability density and carries a restriction over the slope integrations $\{\gamma_X, \gamma_X'\}$. Assuming that the surface is perfectly conducting, the scattering problem is easier to solve because in (7) the polarization term F_{pq} is simpler. The following section explores this aspect.

4. Backscattering coefficient for a perfectly-conducting surface

For a perfectly-conducting surface, the polarization term F_{pq} is expressed from (7) and the backscattering coefficient given by (10) becomes ($\{\sigma_{v_s v_s} = \sigma_{h_s h_s}, \sigma_{v_s h_s} = 0\}$)

$$\sigma_{v_s v_s} = \frac{k^2}{\pi}\int_0^\infty r\,\mathrm{d}r \int_0^{2\pi} \langle\cdots\rangle \exp[\mathrm{j}r(q_x^2+q_y^2)^{1/2}\cos(\Phi-\varphi_s)]\,\mathrm{d}\Phi \tag{13}$$

with

$$\langle\cdots\rangle = \iiiint\!\!\iint [\sin\theta_s(\gamma_x\cos\varphi_s + \gamma_y\sin\varphi_s) - \cos\theta_s] \times[\sin\theta_s(\gamma_x'\cos\varphi_s + \gamma_y'\sin\varphi_s) - \cos\theta_s]\exp[\mathrm{j}q_z(z'-z'')]\,p(\vec{V}_{xy})\,\mathrm{d}\vec{V}_{xy} \tag{13a}$$

where the probability density $p(\vec{V}_{xy})$ of vector $\vec{V}_{xy}^T = [z'z''\gamma_x\gamma_x'\gamma_y\gamma_y']$ is the surface height and slope joint probability density defined in Cartesian coordinates. Since the PDF is known in polar coordinates, the integral has to be determined in polar coordinates. Thus, rotating by Φ, we obtain

$$\begin{aligned} \gamma_x &= \gamma_X\cos\Phi - \gamma_Y\sin\Phi \qquad & \gamma_x' &= \gamma_X'\cos\Phi - \gamma_Y'\sin\Phi \\ \gamma_y &= \gamma_X\sin\Phi + \gamma_Y\cos\Phi \qquad & \gamma_y' &= \gamma_X'\sin\Phi + \gamma_Y'\cos\Phi \end{aligned} \tag{14}$$

and the integral (13*a*) becomes, with the Jacobian equal to one,

$$\langle\cdots\rangle = \iiiint\!\!\iint [\sin\theta_s(\gamma_X c + \gamma_Y s) - \cos\theta_s][\sin\theta_s(\gamma_X' c + \gamma_Y' s) - \cos\theta_s] \times\exp[\mathrm{j}q_z(z'-z'')]\,p(\vec{V}_{XY})\,\mathrm{d}\vec{V}_{XY} \tag{15}$$

with

$$c = \cos(\varphi_s - \Phi) \qquad s = \sin(\varphi_s - \Phi). \tag{15a}$$

Note that the slope integration boundaries of (15) are infinite because the statistical shadowing function is ignored.

Assuming an uncorrelated Gaussian process with surface height z' of variance ω^2, surface slope γ of variance σ^2, and from Smith's [14, 15] monostatic one-dimensional statistical shadowing function given by

$$S(z',\gamma,v) = \Upsilon(\mu-\gamma)\left[1 - \tfrac{1}{2}\,\mathrm{erfc}\left(\frac{z'}{\omega\sqrt{2}}\right)\right]^{\Lambda(v)} \tag{16}$$

with

$$\Lambda(v) = \frac{e^{-v^2} - v\sqrt{\pi}\,\mathrm{erfc}(v)}{2v\sqrt{\pi}} \qquad v = \frac{\mu}{\sqrt{2}\sigma} = \frac{\cot\theta}{\sqrt{2}\sigma} \tag{16a}$$

and

$$\Upsilon(\mu - \gamma) = \begin{cases} 0 & \text{if} \quad \gamma \geqslant \mu = \cot\theta \\ 1 & \text{if} \quad \gamma < \mu. \end{cases} \tag{16b}$$

Bourlier *et al* [12] showed that Smith's monostatic two-dimensional statistical shadowing function is expressed from (16) by replacing γ in γ_X and σ in σ_X given by (12*b*). In [12], the correlation between the slopes and the heights have been investigated on the average shadowing function. Since the difference between the correlated and uncorrelated results is small, the uncorrelated statistical shadowing function is used. This allows for a simpler shadowing function.

4.1. Ensemble average without shadowing effect

When the shadowing function is not included, the probability density is not modified, and the integration boundaries over the slopes $\{\gamma_X, \gamma'_X\}$ are infinite. The ensemble average (15) is then equal to

$$\begin{aligned}\langle\cdots\rangle = \cos^2\theta_s \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \big(&\tan^2\theta_s\big\{c^2 E_4(\gamma_X\gamma'_X) + s^2 E_4(\gamma_Y\gamma'_Y) + cs[E_4(\gamma_X\gamma'_Y) + E_4(\gamma_Y\gamma'_X)]\big\} \\ &- \tan\theta_s\big\{c[E_4(\gamma_X) + E_4(\gamma'_X)] + s[E_4(\gamma_Y) + E_4(\gamma'_Y)]\big\} + 1\big) \\ &\times \exp[\mathrm{j}q_z(z' - z'')]\,\mathrm{d}z'\,\mathrm{d}z''\end{aligned} \tag{17}$$

where $E_4(\cdots)$ denotes the expected value given by

$$E_4(\cdots) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (\cdots)\,p(\vec{V}_{XY})\,\mathrm{d}\gamma_X\,\mathrm{d}\gamma'_X\,\mathrm{d}\gamma_Y\,\mathrm{d}\gamma'_Y. \tag{17a}$$

From (A12) and (A13) we show that

$$c[E_4(\gamma_X) + E_4(\gamma'_X)] + s[E_4(\gamma_Y) + E_4(\gamma'_Y)] = \frac{(z'' - z')(c\sigma_X f_1 + s\sigma_Y f_{16})}{\omega(1 - f_0)}\,p(z', z'') \tag{18}$$

where $p(z', z'')$ is the surface height joint probability density given by (A9). From (B13), (B15), (B17) and (B19), we show that

$$\begin{aligned} c^2 E_4(\gamma_X\gamma'_X) &+ s^2 E_4(\gamma_Y\gamma'_Y) + cs[E_4(\gamma_X\gamma'_Y) + E_4(\gamma_Y\gamma'_X)] \\ &= p(z', z'')\left\{\frac{(c\sigma_X f_1 + s\sigma_Y f_{16})^2}{1 - f_0^2}\left[\frac{(z' f_0 - z'')(z' - z'' f_0)}{\omega^2(1 - f_0^2)} - f_0\right]\right. \\ &\quad \left. +(c^2\sigma_X^2 f_2 + s^2\sigma_Y^2 f_{56} + 2sc\sigma_X\sigma_Y f_{36})\right\} \end{aligned} \tag{19}$$

with

$$\begin{aligned} f_0 &= R_0/\omega^2 & f_1 &= -R_1/(\omega\sigma_X) \\ f_2 &= -R_2/\sigma_X^2 & f_{16} &= -C_{16}/(\omega\sigma_Y) \\ f_{56} &= -C_{56}/\sigma_Y^2 & f_{36} &= -C_{36}/(\sigma_Y\sigma_X). \end{aligned} \tag{19a}$$

Now, the ensemble average (17) requires two integrations over the heights $\{z', z''\}$ based on the calculation of the surface height joint characteristic function χ_1 equal to the Fourier transform of the surface height joint probability density $p(z', z'')$ expressed as

$$\chi_1 = \chi(1) \tag{20}$$

with

$$\chi(\cdots) = \frac{1}{2\pi\omega^2(1-f_0^2)^{1/2}} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (\cdots) \exp\left[-\frac{1}{2\omega^2(1-f_0^2)}(z'^2 + z''^2 - 2f_0 z' z'')\right] \times \exp[\mathrm{j}q_z(z'-z'')\,\mathrm{d}z'\,\mathrm{d}z''. \tag{21}$$

Using the following variable transformations:

$$\begin{aligned} z' &= \omega[Z'(1+f_0)^{1/2} - Z''(1-f_0)^{1/2}] \\ z'' &= \omega[Z'(1+f_0)^{1/2} + Z''(1-f_0)^{1/2}] \end{aligned} \tag{22}$$

χ_1 becomes

$$\chi_1 = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \exp(-Z''^2)\exp[-2\mathrm{j}q_z\omega Z''(1-f_0)^{1/2}]\,\mathrm{d}Z'' = \exp[-q_z^2\omega^2(1-f_0)]. \tag{23}$$

Consequently, from (21) and (23), respectively, we can write

$$\chi(z''-z') = \mathrm{j}\frac{\partial\chi_1}{\partial q_z} = -2\mathrm{j}q_z\omega^2(1-f_0)\chi_1 \tag{23a}$$

and

$$\chi\left(\frac{[z'f_0 - z''][z' - z''f_0]}{\omega^2[1-f_0^2]^2} - \frac{f_0}{1-f_0^2}\right) = -q_z^2\omega^2\chi_1 = -\frac{\partial\chi_1}{\partial f_0}. \tag{23b}$$

Therefore, the use of (18), (19), (23*a*) and (23*b*) leads to the following ensemble average:

$$\langle\cdots\rangle = \cos^2\theta_s[\chi_1 + \mathrm{j}\chi_{PO1}(\sigma_X\tan\theta_s) + \chi_{PO2}(\sigma_X\tan\theta_s)^2] \tag{24}$$

with

$$\begin{aligned} \chi_{PO1} &= \frac{\partial\chi_1}{\partial q_z}\frac{c\sigma_X f_1 + s\sigma_Y f_{16}}{\omega(1-f_0)} = 2q_z\omega\chi_1\left(cf_1 + \frac{s\sigma_Y f_{16}}{\sigma_X}\right) \\ \chi_{PO2} &= \chi_1\left(c^2 f_2 + \frac{s^2\sigma_Y^2 f_{56}}{\sigma_X^2} + \frac{2sc\sigma_Y f_{36}}{\sigma_X}\right) - \frac{\partial\chi_1}{\partial f_0}\left(cf_1 + \frac{s\sigma_Y f_{16}}{\sigma_X}\right)^2 \\ &= \chi_1\left[c^2 f_2 + \frac{s^2\sigma_Y^2 f_{56}}{\sigma_X^2} + \frac{2sc\sigma_Y f_{36}}{\sigma_X} - q_z\omega^2\left(cf_1 + \frac{s\sigma_Y f_{16}}{\sigma_X}\right)^2\right] \end{aligned} \tag{24a}$$

where χ_1 corresponds to the obtained term when the stationary phase method is used, and $\{\chi_{PO1}, \chi_{PO2}\}$ characterize the functions introduced by the physical optics approach.

For an isotropic surface, $R_{02} = 0$, which means from (12*b*) and (12*c*) that $\{C_{16} = 0, C_{36} = 0, \sigma_x = \sigma_y\}$, i.e. $\{f_{16} = 0, f_{36} = 0, \sigma_X = \sigma_Y\}$, and (24*a*) becomes

$$\begin{aligned} \chi_{PO1} &= 2c\chi_1 q_z\omega f_1 \\ \chi_{PO2} &= \chi_1[(c^2 f_2 + s^2 f_{56}) - c^2(q_z\omega f_1)^2]. \end{aligned} \tag{25}$$

For the Gaussian and power height autocorrelation functions R_0 expressed in table 1, with height variance ω^2, with a length correlation L_c, the $\{f_0, f_1, f_{56}, f_2\}$ functions are given in table 1 with respect to $u = r/L_c$. Substituting these equations into (25), the $\{\chi_1, \chi_{PO1}, \chi_{PO2}\}$ functions are obtained with respect to $\{\delta = q_z\omega, u\}$. It can be noted that the contribution of $\{\chi_{PO1}, \chi_{PO2}\}$ terms is negligible, if $\sigma_X\tan\theta_s \ll 1$, i.e. when the surface RMS slope σ_X is much smaller than the slope incident beam $\mu = \cot\theta_s$.

Table 1. Derivations of the functions $\{f_0, f_1, f_{56}, f_2, |f_0''(0)|\}$ for Gaussian and power one-dimensional surface height autocorrelation functions $R_0(r)$.

Definitions	Gaussian function	Power function
$R_0(r)$	$\omega^2 \exp\left(-\frac{pr^2}{L_c^2}\right)$	$\omega^2/\left(1+\frac{r^2}{L_c^2}\right)^p$
$f_0(u) = \frac{R_0(r)}{\omega^2}$ with $u = \frac{r}{L_c}$	$\exp(-pu^2)$	$\frac{1}{(1+u^2)^p}$
$f_1(u) = \frac{-1}{\omega\sigma_X}\frac{dR_0}{dr}\Big\vert_{r=uL_c} = \frac{-1}{\|f_0''(0)\|^{1/2}}\frac{df_0}{du}$	$u\sqrt{2p}\exp(-pu^2)$	$\frac{u\sqrt{2p}}{(1+u^2)^{p+1}}$
$f_{56}(u) = \frac{-1}{r\omega\sigma_X}\frac{dR_0}{dr}\Big\vert_{r=uL_c} = \frac{-1}{u\|f_0''(0)\|^{1/2}}\frac{df_0}{du}$	$\sqrt{2p}\exp(-pu^2)$	$\frac{\sqrt{2p}}{(1+u^2)^{p+1}}$
$f_2(u) = \frac{-1}{\sigma_X^2}\frac{d^2R_0}{dr^2}\Big\vert_{r=uL_c} = \frac{-1}{\|f_0''(0)\|}\frac{d^2f_0}{du^2}$	$\exp(-pu^2)(1-2pu^2)$	$\frac{1-u^2(1+2p)}{(1+u^2)^{p+2}}$
$\|f_0''(0)\| = \frac{\sigma_X^2L_c^2}{\omega^2}$	$2p$	$2p$

4.2. *Ensemble average with shadowing effect*

The shadowing effect introduced a restriction on the integrations over the slopes $\{\gamma_X, \gamma_X'\}$, whereas the range integrations over the slopes $\{\gamma_Y, \gamma_Y'\}$ remain $\{]-\infty; \infty[,]-\infty; \infty[\}$. From (16*b*) and according to Bourlier *et al*'s work [12] (extension of the one-dimensional shadowing function to the two-dimensional shadowing function), the range of integrations over $\{\gamma_X, \gamma_X'\}$ becomes $\{]-\infty; \mu],]-\infty; \mu]\}$, with $\mu = \cot\theta_s$ the incident beam slope. Moreover, the surface height probability density is modified by the second term of (16) in square brackets.

When the shadowing function is ignored, the ensemble average is given by (24) and (24*a*). It depends on the χ_1 surface height characteristic function expressed from (23), its derivatives according to $\{q_z, f_0\}$, and expected values $\{f_1, f_{16}, f_2, f_{36}, f_{56}\}$ of the $[C_{XY}]$ (12) covariance matrix. Since the shadowing function modified the surface height probability (11), surface height and slope joint probability density $p_S(\vec{V}_{XY})$ with the shadowing effect of the vector $\vec{V}_{XY}^T = [z'z''\gamma_X\gamma_X'\gamma_Y\gamma_Y']$ has to be determined.

For an uncorrelated Gaussian process with $\{\omega = 1, \sigma_X = 1, \sigma_Y = 1\}$, we have $\{R_0, R_1, R_2, C_{16}, C_{36}, C_{56}, \sigma_{XY}\} = 0$ in (12) covariance matrix $[C_{XY}]$, which involves, from (11) and (16), that the surface height $\{\xi', \xi''\}$ and slope $\{\zeta_X, \zeta_X', \zeta_Y, \zeta_Y'\}$ uncorrelated probability density with shadowing effect is written as

$$p_S(\vec{V}_{DXY}) = \frac{1}{(2\pi)^3}\exp\left(-\frac{\xi'^2}{2} - \frac{\xi''^2}{2} - \frac{\zeta_X^2}{2} - \frac{\zeta_X'^2}{2} - \frac{\zeta_Y^2}{2} - \frac{\zeta_Y'^2}{2}\right) \times\left[1 - \frac{1}{2}\operatorname{erfc}\left(\frac{\xi'}{\sqrt{2}}\right)\right]^{\Lambda}\left[1 - \frac{1}{2}\operatorname{erfc}\left(\frac{\xi''}{\sqrt{2}}\right)\right]^{\Lambda} \tag{26}$$

where $\vec{V}_{DXY}^T[\xi'\xi''\zeta_X\zeta_X'\zeta_Y\zeta_Y']$ denotes the transpose uncorrelated vector of $\vec{V}_{XY}^T$.

The correlation is introduced by writing

$$\vec{V}_{XY} = [C_{XY}]^{1/2}\vec{V}_{DXY} \tag{27}$$

due to the fact that

$$E[\vec{V}_{XY}\vec{V}_{XY}^T] = E([C_{XY}]^{1/2}\vec{V}_{DXY}\vec{V}_{DXY}^T[C_{XY}]^{H/2}) = [C_{XY}] \tag{27a}$$

where H is the conjugate transpose. Since the samples $\{\xi', \xi'', \zeta_X, \zeta'_X, \zeta_Y, \zeta'_Y\}$ are independent, the expected value $E(\vec{V}_{DXY}\vec{V}^T_{DXY})$ is equal to the identity matrix. For computing the matrix $[C_{XY}]^{1/2}$, we can write

$$[C_{XY}] = [V][\Sigma][V]^T \tag{28}$$

where $[V]$ is the unitary eigenvectors matrix of $[C_{XY}]$, $[\Sigma]$ is the eigenvalues matrix of $[C_{XY}]$.

$[\Sigma]$ being diagonal implies that

$$[C_{XY}]^n = [V][\Sigma^n][V]^T. \tag{29}$$

If we can calculate the eigenvalues and the eigenvectors of the matrix $[C_{XY}]$, by inverting (27), this leads to $\vec{V}_{DXY} = [C_{XY}]^{-1/2}\vec{V}_{XY}$. The components $\{\xi', \xi'', \zeta_X, \zeta'_X, \zeta_Y, \zeta'_Y\}$ are then expressed according to $\{z', z'', \gamma_X, \gamma'_X, \gamma_Y, \gamma'_Y\}$. Substituting them into (26), the probability density with shadowing effect $p_S(\vec{V}_{XY})$ is determined by

$$p_S(\vec{V}_{XY}) = p_S(\vec{V}_{DXY})\frac{\mathrm{d}\vec{V}_{DXY}}{\mathrm{d}\vec{V}^T_{XY}} = \frac{p_S(\vec{V}_{DXY})}{|[C_{XY}]|^{1/2}}. \tag{30}$$

Unfortunately, the analytical determinations of the eigenvalues and eigenvectors of the covariance matrix $[C_{XY}]$ are very difficult for a six-dimensional matrix. Moreover, the scattering coefficient with shadowing effect obtained from the ensemble average (17), requires integrations over the slopes $\{\gamma_X, \gamma'_X, \gamma_Y, \gamma'_Y\}$ with range limits of $\{]-\infty; \mu],]-\infty; \mu],]-\infty; \infty[,]-\infty; \infty[\}$ and the integration over the heights multiplied by the exponential term $\exp[\mathrm{j}q_z(z' - z'')]$.

To solve the problem analytically, the cross-correlation between the heights and the slopes quantified by $\{R_1, C_{16}\}$ in (12) is assumed to be negligible, leading to the following covariance matrix:

$$[C_{XY}] = \begin{bmatrix} [H] & [0] \\ [0] & [S] \end{bmatrix} \quad \text{with} \quad \begin{cases} [H] = \begin{bmatrix} \omega^2 & R_0 \\ R_0 & \omega^2 \end{bmatrix} \\ [S] = \begin{bmatrix} \sigma_X^2 & -R_2 & \sigma_{XY}^2 & -C_{36} \\ -R_2 & \sigma_X^2 & -C_{36} & \sigma_{XY}^2 \\ \sigma_{XY}^2 & -C_{36} & \sigma_Y^2 & -C_{56} \\ -C_{36} & \sigma_{XY}^2 & -C_{56} & \sigma_Y^2 \end{bmatrix} \end{cases} \tag{31}$$

where $\{[H], [S]\}$ are the height and slope covariance matrices, respectively. From (24*a*), this assumption is valid from small values of $q_z\omega$ with $q_z = 2k\cos\theta_s$ (ω is the surface RMS elevation), i.e. for a rough surface with $4\pi\cos\theta_s\omega \ll \lambda$.

Since $\{R_1, C_{16}\} = 0 \Rightarrow \{f_1, f_{16}\} = 0$, and from (24) and (24*a*), the ensemble average with shadowing effect can be expressed as

$$\langle\cdots\rangle_S = \cos^2\theta_s[\chi_{S1}E_{S4}(1) + (\sigma_X\tan\theta_s)^2\chi_{SPO2}] \tag{32}$$

with

$$\chi_{SPO2} = -\frac{\chi_{S1}}{\sigma_X^2}[c^2E_{S4}(\gamma_X\gamma'_X) + s^2E_{S4}(\gamma_Y\gamma'_Y) + 2scE_{S4}(\gamma_X\gamma'_Y)] \tag{32a}$$

and from (C8)

$$\chi_{S1} = |F(q_z\omega\sqrt{2}(1 - f_0)^{1/2})|^2 \tag{32b}$$

where the integral function $F(\cdots)$ given by (C9) is the surface height characteristic function equal to the Fourier transform of the surface height probability density modified by the shadow. If the shadow is ignored then $\chi_{S1} = \chi_1$ (23). Noting that $\chi_{SPO1} = 0$, the expected value $E_{S4}(\cdots)$ with shadowing effect is defined as

$$E_{S4}(\cdots) = \int_{-\infty}^{\mu} \mathrm{d}\gamma_X \int_{-\infty}^{\mu} \mathrm{d}\gamma'_X \int_{-\infty}^{\infty} \mathrm{d}\gamma_Y \int_{-\infty}^{\infty} (\cdots) p(\gamma_X, \gamma'_X, \gamma_Y, \gamma'_Y)\, \mathrm{d}\gamma'_Y \quad (33)$$

where the slope surface joint probability density $p(\gamma_X, \gamma'_X, \gamma_Y, \gamma'_Y)$ is characterized by the covariance matrix $[S]$ (31). When the shadow is not included, involving $\mu \to \infty$, we obtain $E_{S4}(1) = 1$, $E_{S4}(\gamma_X\gamma'_X) = -R_2 = \sigma_X^2 f_2$, $E_{S4}(\gamma_Y\gamma'_Y) = -C_{56} = \sigma_Y^2 f_{56}$, $E_{S4}(\gamma_X\gamma'_Y) = -C_{36} = \sigma_X\sigma_Y f_{56}$, and (32) $\langle\cdots\rangle_S$ is equal to (24) $\langle\cdots\rangle$ with $\{\chi_{S1} = \chi_1, f_1 = 0, f_{16} = 0\}$.

From (D6), (D12) and (D18), we find

$$\begin{aligned} -[c^2 E_{4S}(\gamma_X\gamma'_X) + s^2 E_{4S}(\gamma_Y\gamma'_Y) + 2sc E_{4S}(\gamma_X\gamma'_Y)] \\ = \tfrac{1}{2}[1 + \mathrm{erf}(\mu_0)][c^2\sigma_X^2 f_2 + s^2\sigma_Y^2 f_{56} + 2cs\sigma_X\sigma_Y f_{36}] \\ - \frac{\mu \exp(-\mu_0^2)}{2\sqrt{\pi}} \frac{[c^2\sigma_X^2(1 + f_2) + s(\sigma_{XY}^2 + \sigma_X\sigma_Y f_{36})]^2}{\sigma_X^3(1 + f_2)^{3/2}}. \end{aligned} \quad (33a)$$

Substituting (32*a*), (32*b*), (33*a*) and (D5) into (32), the ensemble average with shadowing effect becomes

$$\begin{aligned} \langle\cdots\rangle_S = \cos^2\theta_s \left|F(q_z\omega\sqrt{2}[1 - f_0]^{1/2})\right|^2 \\ \times\left\{\varepsilon_1\left[1 + (\sigma_X\tan\theta_s)^2\left(c^2 f_2 + s^2\frac{\sigma_Y^2 f_{56}}{\sigma_X^2} + 2cs\frac{\sigma_Y f_{36}}{\sigma_X}\right)\right]\right. \\ \left. - \frac{\varepsilon_2\tan\theta_s}{\sigma_X(1 + f_2)^{3/2}}\left[c\sigma_X(1 + f_2) + s\left(\frac{\sigma_{XY}^2}{\sigma_X} + \sigma_Y f_{36}\right)\right]^2\right\} \end{aligned} \quad (34)$$

with

$$\varepsilon_1 = \tfrac{1}{2}[1 + \mathrm{erf}(\mu_0)] \qquad \varepsilon_2 = \exp(-\mu_0^2)/(2\sqrt{\pi}) \qquad \mu_0 = \frac{\mu}{\sigma_X(1 + f_2)^{1/2}}. \quad (34a)$$

If the shadowing function is ignored then $\{\Lambda = 0, \mu_0 \to \infty\}$ involving that $\{\varepsilon_1 = 1, \varepsilon_2 = 0\}$ and $|F|^2 = \exp[-q_z^2\omega^2(1 - f_0)]$. The ensemble average becomes $\langle\cdots\rangle_S = \cos^2\theta_s\chi_1[1+(\sigma_X\tan\theta_s)^2\chi_{PO2}]$, then (24) is found with $\{f_1 = 0, f_{16} = 0\}$ since the correlation between the slopes and heights is neglected.

For an isotropic surface, $R_{02} = 0$, which means from (12*b*) and (12*c*) that $\{C_{16} = 0, C_{36} = 0, \sigma_x = \sigma_y\}$, i.e. $\{f_{16} = 0, f_{36} = 0, \sigma_X = \sigma_Y\}$, and (34) becomes

$$\begin{aligned} \langle\cdots\rangle_S = \cos^2\theta_s\left|F(q_z\omega\sqrt{2}(1 - f_0)^{1/2})\right|^2\{\varepsilon_1[1 + (\sigma_X\tan\theta_s)^2(c^2 f_2 + s^2 f_{56})] \\ -\varepsilon_2\tan\theta_s c^2\sigma_X(1 + f_2)^{1/2}\}. \end{aligned} \quad (35)$$

5. Simulations of the backscattering coefficient with and without shadow

This section presents the backscattering coefficients given by (13) with and without shadow obtained from the ensemble average calculated in the previous section, for an isotropic perfectly-conducting surface. The results are compared with the stationary phase and geometrical optics approximations which can be defined as particular cases of Kirchhoff's solution. The model is also compared with respect to the choice of the surface height autocorrelation function.

5.1. Development

Substituting (24) into (13) with (15*a*) and (25), for an isotropic surface, the backscattering coefficient without shadowing effect (exponent US) with $u = r/L_c$ (L_c denotes the surface correlation length) is

$$\sigma^{US} = \frac{L_c^2 k^2 \cos^2\theta_s}{\pi} \int_{-\infty}^{\infty} u\,\mathrm{d}u \int_0^{2\pi} \exp[-q_z^2\omega^2(1 - f_0)] \exp[\mathrm{j}r(q_x^2 + q_y^2)^{1/2}\cos(\Phi - \varphi_s)]$$
$$\times \left\{ 1 + \frac{(\sigma_X \tan\theta_s)^2[f_2 + f_{56} - (q_z\omega f_1)^2]}{2} + 2\mathrm{j}(\sigma_X \tan\theta_s)(q_z\omega f_1)\cos(\Phi - \varphi_s) \right.$$
$$\left. + \frac{\cos[2(\Phi - \varphi_s)](\sigma_X \tan\theta_s)^2[f_2 - (q_z\omega f_1)^2 - f_{56}]}{2} \right\} \mathrm{d}\Phi. \quad (36)$$

Knowing that

$$\Psi_0 = \int_0^{2\pi} \exp[\mathrm{j}x\cos(\Phi - \varphi_s)]\,\mathrm{d}\Phi = 2\pi J_0(x) \quad (36a)$$

we have

$$\Psi_1 = \int_0^{2\pi} \cos(\Phi - \varphi_s)\exp[\mathrm{j}x\cos(\Phi - \varphi_s)]\,\mathrm{d}\Phi = -\mathrm{j}\frac{\partial\Psi_0}{\partial x} = 2\pi\mathrm{j}J_1(x)$$
$$\Psi_2 = \int_0^{2\pi} [\cos(\Phi - \varphi_s)]^2\exp[\mathrm{j}x\cos(\Phi - \varphi_s)]\,\mathrm{d}\Phi = -\mathrm{j}\frac{\partial\Psi_1}{\partial x} = \pi[J_0(x) - J_2(x)] \quad (36b)$$
$$\Psi_3 = \int_0^{2\pi} \cos[2(\Phi - \varphi_s)]\exp[\mathrm{j}x\cos(\Phi - \varphi_s)]\,\mathrm{d}\Phi = 2\Psi_2 - \Psi_0 = -2\pi J_2(x)$$

where J_i is the ith-order Bessel function. Since the autocorrelation function is independent of the direction Φ, and substituting (36*a*) and (36*b*) into (36), the integration over Φ leads to

$$\sigma^{US} = (k_c\cos\theta_s)^2 \int_0^{\infty} u\exp[-q_z^2\omega^2(1 - f_0)] \left\{ \left[1 + \frac{(\sigma_X\tan\theta_s)^2[f_2 + f_{56} - (q_z\omega f_1)]}{2} \right] \right.$$
$$\times J_0(\sqrt{2}uk_c\sin\theta_s) - 2(\sigma_X\tan\theta_s)(q_z\omega f_1)J_1(\sqrt{2}uk_c\sin\theta_s)$$
$$\left. - \frac{(\sigma_X\tan\theta_s)^2[f_2 - (q_z\omega f_1)^2 - f_{56}]}{2} J_2(\sqrt{2}uk_c\sin\theta_s) \right\} \mathrm{d}u \quad (37)$$

with $k_c = \sqrt{2}kL_c$. For any surface height autocorrelation function $R_0(r)$ we can write

$$\sigma_X^2 = -\left.\frac{\partial^2 R_0}{\partial r^2}\right|_{r=0} = -\left.\frac{\omega^2}{L_c^2}\frac{\partial^2 f_0}{\partial u^2}\right|_{u=0} \quad\Rightarrow\quad \omega = \frac{\sigma_X L_c}{|f_0''(0)|^{1/2}}. \quad (38)$$

Equation (38) involves $q_z\omega = (2/|f_0''(0)|)^{1/2}k_c\cos\theta_s\,\sigma_X$.

The functions $\{f_0, f_1, f_2, f_{56}, |f_0''(0)|\}$ for Gaussian and power surface height autocorrelation functions of parameter p are expressed in table 1 with respect to u, and plotted in figure 2. It is interesting to study these autocorrelation functions because they have the same slope variance and are used in the literature [10].

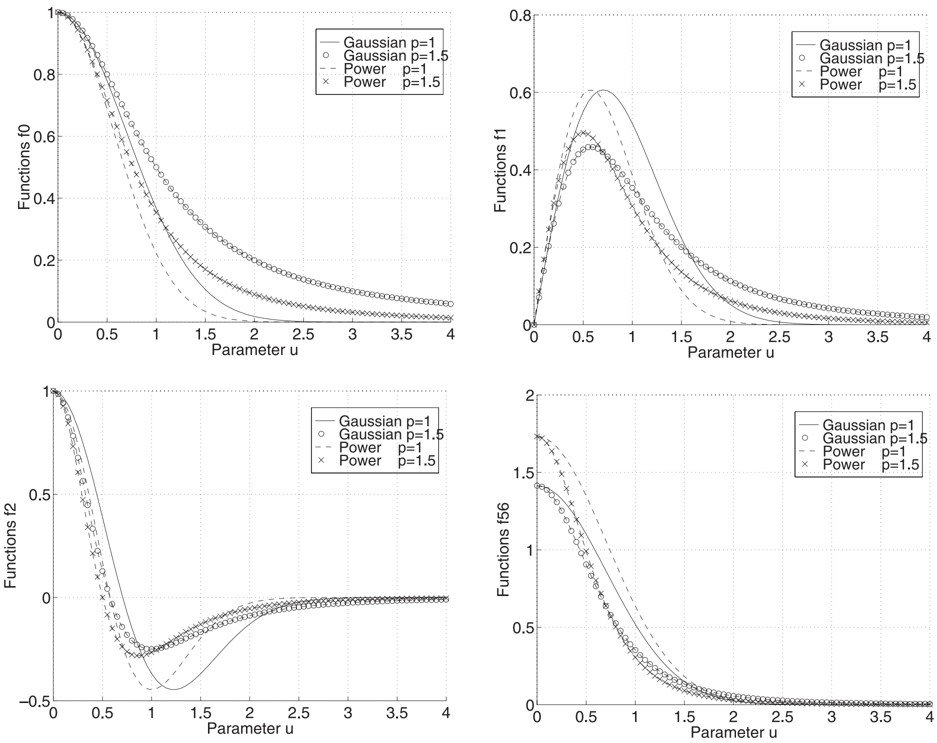

Figure 2. Functions $\{f_0, f_1, f_2, f_{56}\}$ versus $u = r/L_c$ for Gaussian and power surface height autocorrelation functions with $p = 1$ and 1.5.

From (35), and using the same method, the scattering coefficient with shadow (exponent S) is expressed as

$$\sigma^S = (k_c \cos\theta_s)^2 \int_0^\infty u|F(q_z\omega\sqrt{2}(1-f_0)^{1/2})|^2 \times \left\{ \frac{\varepsilon_1[2 + (\sigma_X \tan\theta_s)^2(f_2 + f_{56})] - \varepsilon_2 \tan\theta_s \sigma_X (1+f_2)^{1/2}}{2} J_0(\sqrt{2}uk_c \sin\theta_s) - \frac{\varepsilon_1(\sigma_X \tan\theta_s)^2(f_2 - f_{56}) - \varepsilon_2 \tan\theta_s \sigma_X (1+f_2)^{1/2}}{2} J_2(\sqrt{2}uk_c \sin\theta_s) \right\} \mathrm{d}u \quad (39)$$

with $q_z\omega\sqrt{2}(1-f_0)^{1/2} = 2k_c \cos\theta_s\, \sigma_X (1-f_0)^{1/2}/|f_0''(0)|^{1/2}$. As $\{f_0, f_1, f_2, f_{56}\}$, $\{F, \varepsilon_1, \varepsilon_2\}$ also depends on u.

The stationary phase approximation (index SP) assumes that the electromagnetic field is scattered around the specular direction, which means in (15) that $\{\gamma_X c, \gamma_X' c\} = -\tan\theta_s$ corresponding to the orthogonal direction of incident beam $\mu = \cot\theta_s$, and $\{\gamma_Y s, \gamma_Y' s\} = 0$. Therefore, the ensemble average does not depend on the slopes and becomes

$$\langle\cdots\rangle = \frac{1}{\cos^2\theta_s} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \exp[\mathrm{j}q_z(z' - z'')]p(z', z'')\,\mathrm{d}z'\,\mathrm{d}z'' = \frac{\exp[-q_z^2\omega^2(1-f_0)]}{\cos^2\theta_s}. \quad (40)$$

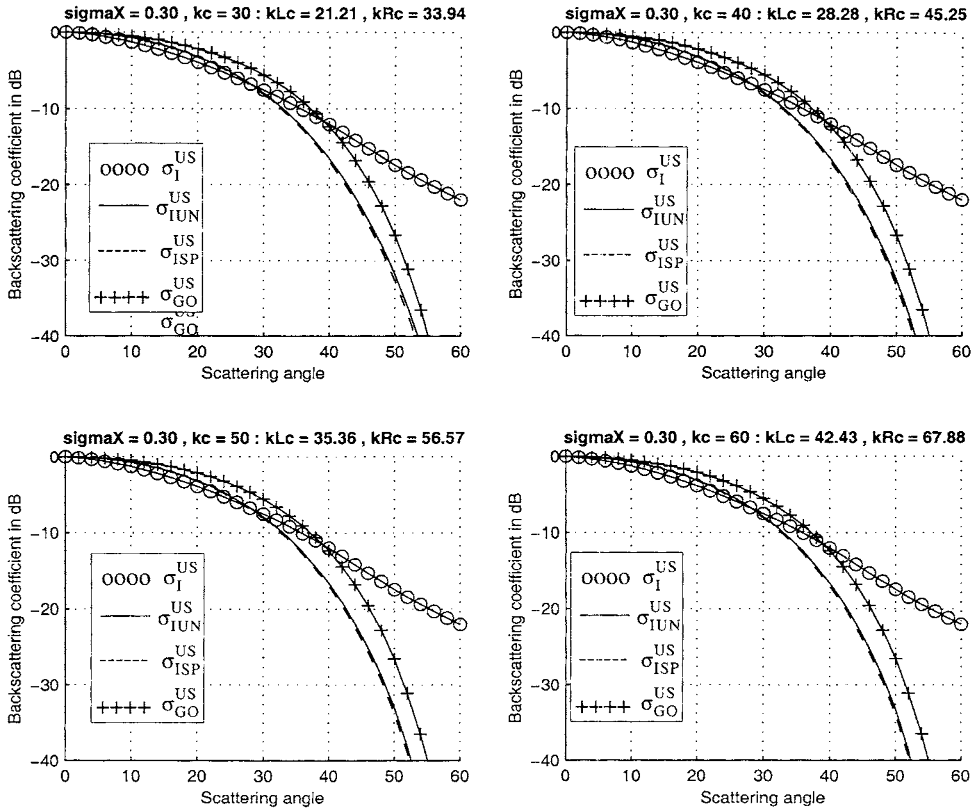

Figure 3. Normalized incoherent backscattering coefficients $\{\sigma_I^{US}, \sigma_{IUN}^{US}, \sigma_{ISP}^{US}, \sigma_{GO}^{US}\}$ in dB without shadowing effect versus the scattering angle θ_s for $\sigma_X = 0.3$ and $k = \{30, 40, 50, 60\}$ with a Gaussian surface height autocorrelation function ($p = 1$). σ^{US}, the circle curve; σ_{UN}^{US}, full curve; σ_{SP}^{US}, broken curve and σ_{GO}^{US}, cross curve.

Substituting (40) into (13) and performing the integration over Φ, we obtain

$$\sigma_{SP}^{US} = \frac{k_c^2}{\cos^2\theta_s} \int_0^\infty u J_0(\sqrt{2} u k_c \sin\theta_s) \exp[-q_z^2 \omega^2 (1 - f_0)] \, \mathrm{d}u. \tag{41}$$

The geometrical optics approximation (index GO) or high-frequency limit is obtained by approximating f_0 by the first two terms of its Taylor series expansion about the origin. $1 - f_0$ then becomes $1 - |f_0''(0)|u^2/2 + \mathrm{O}(u^4)$ in the integral (41) and the integration over u leads to

$$\sigma_{GO}^{US} = \frac{1}{2\sigma_X^2 \cos^4\theta_s} \exp\left(-\frac{\tan^2\theta_s}{2\sigma_X^2}\right). \tag{42}$$

Consequently, σ_{GO}^{US} depends only on the surface slope variance σ_X and on the scattering angle θ_s.

5.2. Simulations of the incoherent backscattering coefficient

The Kirchhoff approximation is valid if $\{kL_c > 2\pi, kR_c > 2\pi\}$, where k is the wavenumber and R_c is the surface mean curvature radius which is equal for Gaussian and power surface

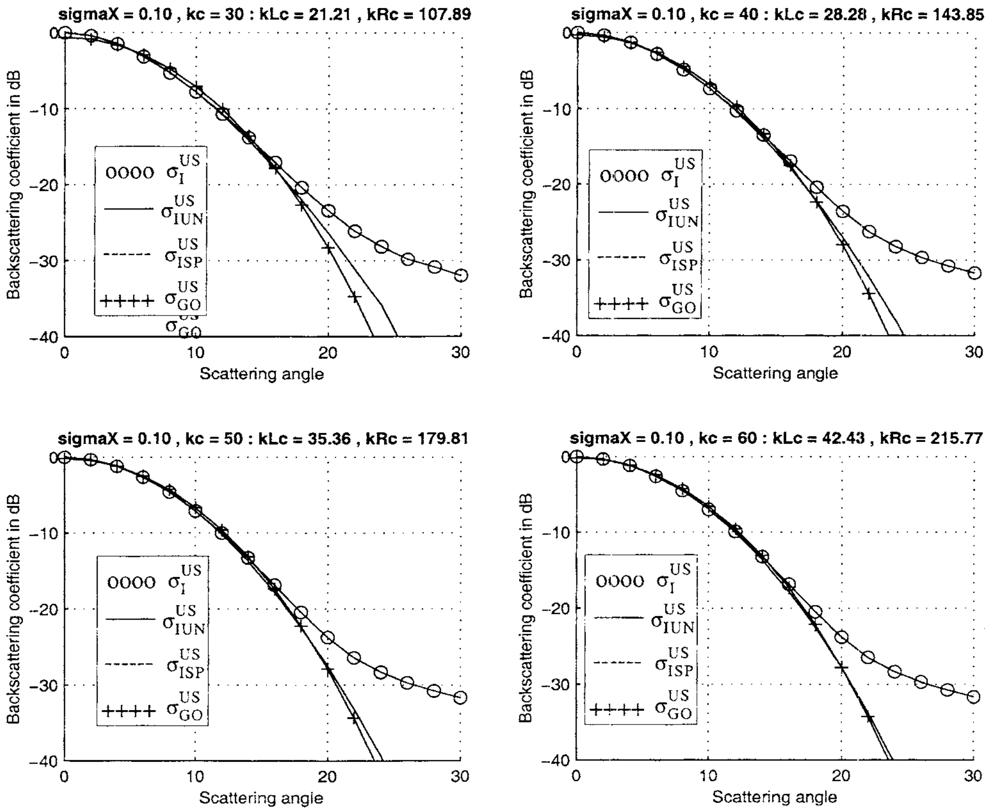

Figure 4. Same simulations as in figure 3 but with $\sigma_X = 0.1$.

height autocorrelation functions with $p = 1$ and small slope $\sigma_X = \sqrt{2}\omega/L_c$ [17]

$$kR_{cG} = \frac{kL_c}{1.95\sigma_X(1 + 3\sigma_X^2/4)} \qquad kR_{cL} = \frac{kL_c}{2.76\sigma_X(1 + 3\sigma_X^2/4)}. \tag{43}$$

In the literature [1] the condition $kR_c = 2\pi$ is replaced either by $kR_c \cos\theta_s > 2\pi$ or $kR_c(\cos\theta_s)^3 > 2\pi$, which is the most often quoted restriction on the applicability of the Kirchhoff theory.

The scattered intensities from a random surface can, in general, be decomposed into coherent and incoherent components. The coherent component σ_{pqC} mostly contributes in the specular direction, whereas the incoherent component σ_{pqI} contributes in all directions, and we can write [18]

$$\sigma_{pqI} = \sigma_{pq} - \sigma_{pqC} \tag{44}$$

where the coherent component σ_{pqC} is calculated from averaging $|\langle\cdots\rangle_C|^2$ defined as

$$|\langle\cdots\rangle_C|^2 = \left| \iiint [\sin\theta_s(\gamma_X c + \gamma_Y s) - \cos\theta_s] \exp(\mathrm{j}q_z z') p(z', \gamma_X, \gamma_Y)\, \mathrm{d}z'\, \mathrm{d}\gamma_X\, \mathrm{d}\gamma_Y \right|^2 \tag{45}$$

with

$$\sigma_{pqC} = \frac{4\pi R_0^2 |\langle E_{pq}^S\rangle|^2}{A_0|E_0|^2}. \tag{45a}$$

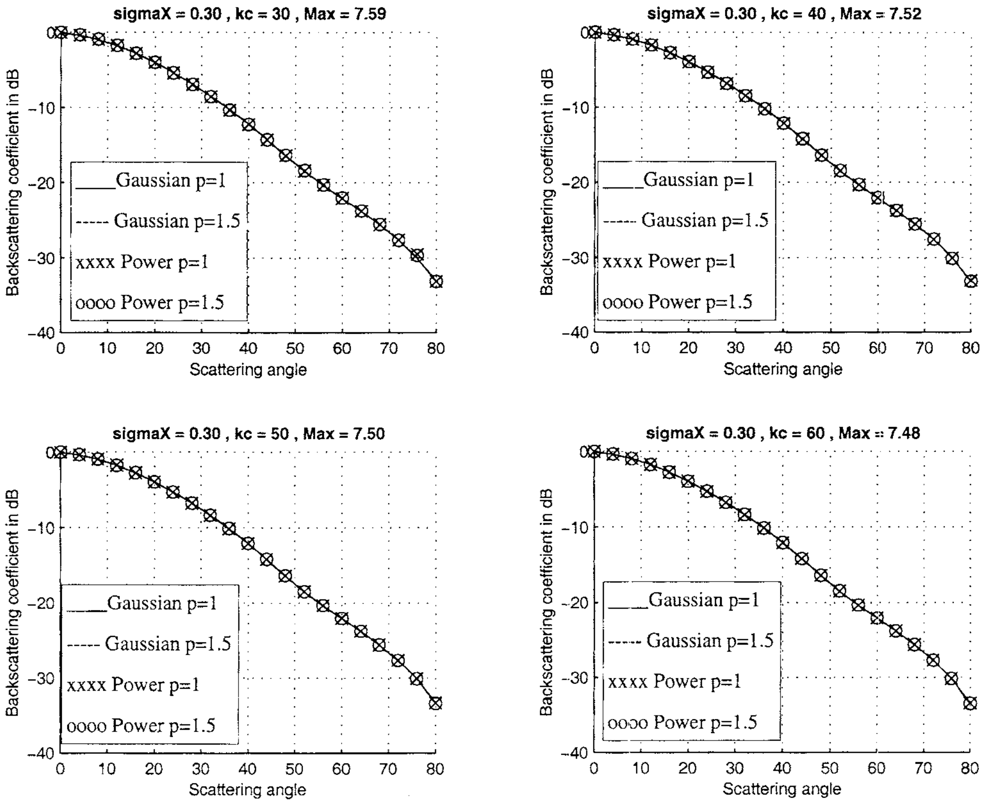

Figure 5. Normalized incoherent backscattering coefficient σ_I^{US} in dB versus the scattering angle θ_s with Gaussian (full curve with $p = 1$, broken curve with $p = 1.5$) and power (circle curve with $p = 1$, cross curve with $p = 1.5$) surface height autocorrelation functions. The parameters are the same as in figure 3.

From (15), we can show that

$$|\langle\cdots\rangle_C|^2 = \langle\cdots\rangle \qquad \text{if} \quad \{f_0, f_1, f_2, f_{16}, f_{36}, f_{56}\} = 0 \tag{46}$$

which is similar to neglecting the correlation. Consequently, from (37), (39), (41), the coherent components without shadow σ_C^{US}, with shadow σ_C^S, and under the stationary phase approximation σ_{CSP}^{US} are expressed as

$$\sigma_C^{US} = (k_c \cos\theta_s)^2 \exp(-q_z^2\omega^2) \int_0^\infty u J_0(\sqrt{2}uk_c \sin\theta_s)\,\mathrm{d}u \tag{46a}$$

$$\sigma_C^S = (k_c \cos\theta_s)^2 \frac{|F(q_z\omega\sqrt{2})|^2}{2} \int_0^\infty u\,\mathrm{d}u\big\{J_0(\sqrt{2}uk_c \sin\theta_s)(2\varepsilon_1 - \varepsilon_2 \tan\theta_s\sigma_X)$$
$$+J_2(\sqrt{2}uk_c \sin\theta_s)\varepsilon_2 \tan\theta_s\sigma_X\big\} \qquad \text{with} \quad \mu_0 = \mu/\sigma_X \text{ in } \{\varepsilon_1, \varepsilon_2\} \tag{46b}$$

$$\sigma_{CSP}^{US} = \sigma_C^{US}. \tag{46c}$$

For the simulations, $k_c = \sqrt{2}kL_c$ is chosen such that the criteria $\{kL_c > 2\pi, kR_c > 2\pi\}$ are valid.

In figures 3 and 4, equation (44), the normalized incoherent backscattering coefficients $\{\sigma_I^{US}, \sigma_{ISP}^{US}, \sigma_{GO}^{US}\}$ ((37) minus (46*a*), circle curve; (41) minus (46*c*), broken curve; (42), cross

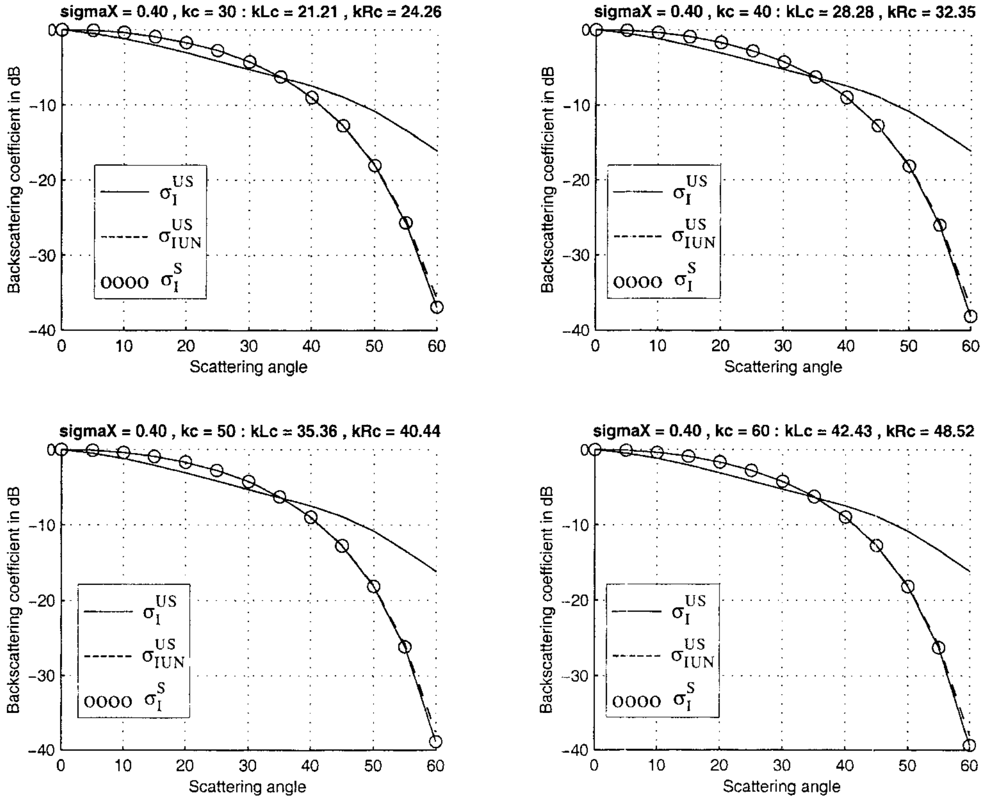

Figure 6. Normalized incoherent scattering coefficients σ_I^{US} (full curve), σ_{IUN}^{US} (broken curve) and σ_I^S (circle curve) in dB versus the scattering angle θ_s for $\sigma_X = 0.4$ and $k_c = \{30, 40, 50, 60\}$ with a Gaussian surface height autocorrelation function.

curve, respectively) and $\sigma_{IUN}^{US} = \sigma_I^{US}$ with $f_1 = 0$ are plotted versus the scattering angle θ_s. $k_c = \{30, 40, 50, 60\}$, $\sigma_X = \{0.3, 0.1\}$, with a Gaussian surface height autocorrelation function ($p = 1$). The incoherent scattering coefficients are normalized by the maximum of σ_I^{US}. As expected, the backscattering curve drops off more slowly with increasing angle as the surface RMS slope increases. If the surface RMS slope σ_X decreases, σ_{IUN}^{US} is similar to σ_{ISP}^{US}, and σ_{ISP}^{US} tends to σ_{GO}^{US}. The deviation between σ_I^{US} and σ_{IUN}^{US} increases with the scattering angle, and σ_I^{US} is larger than σ_{IUN}^{US}. Consequently, when the correlation between the slopes and the heights is ignored, the incoherent component is smaller. This underestimation, which is not noticeable in relation to k_c, decreases when the surface slope decreases, because the $\sigma_X \tan\theta_s$ term in equation (37) becomes smaller.

In figure 5, the normalized incoherent backscattering coefficient σ_I^{US} in dB is represented versus the scattering angle θ_s for $\sigma_X = 0.3$ and $k_c = \{30, 40, 50, 60\}$ with Gaussian (full curve with $p = 1$, broken curve with $p = 1.5$) and power (circle curve with $p = 1$, cross curve with $p = 1.5$) surface height autocorrelation functions. Although the functions $\{f_0, f_1, f_2, f_{56}\}$ are different according to the autocorrelation function (see figure 2), the incoherent backscattering coefficient does not vary with the autocorrelation function. This behaviour may be explained by the fact that the parameter k_c is sufficiently large that the functions $\{f_0, f_1, f_2, f_{56}\}$ can be approximated by the first two terms of its Taylor series expansion about the origin. This

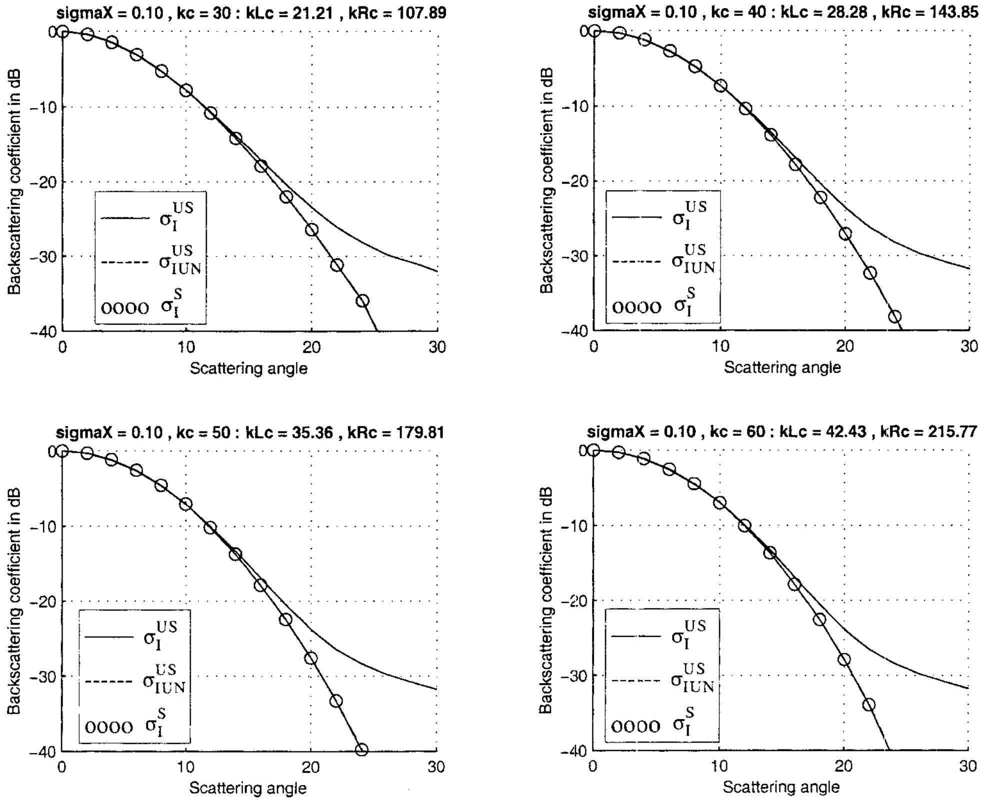

Figure 7. Same variation as in figure 6 with $\sigma_X = 0.1$.

involves the functions becoming the same, because they have the same series development. It is interesting to note that we have a generalization of the geometrical optics. Simulations with $\sigma_X = 0.1$ give the same behaviour.

In figures 6 and 7, the normalized incoherent scattering coefficients σ_I^{US} (full curve), σ_{IUN}^{US} (broken curve), σ_I^S (circle curve (39) minus (46*b*)) in dB are plotted versus the scattering angle θ_s for $\sigma_X = \{0.4, 0.1\}$ and $k_c = \{30, 40, 50, 60\}$ with a Gaussian surface height autocorrelation function. As depicted in figures 6 and 7, σ_{IUN}^{US} is in agreement with σ_I^S. This means that the shadowing effect may be ignored due to the fact that the backscattering coefficient is very small when the shadowing function becomes important. Indeed, the parameter (16*a*) Λ which characterizes the shadowing effect according to the heights, varies between [0; 0.0533] for $\{\sigma_X = 0.4, \theta_s \in [0; 60]^\circ\}$, which implies with (C9) that $[1 - \mathrm{erfc}(X/2)]^\Lambda$ remains equal to one. Moreover, the parameter μ_0 in (34*a*) with $1/\sigma_X \tan\theta_s \in [1.443; \text{infinity}]$ is so large that the $\{\varepsilon_1, \varepsilon_2\}$ functions corresponding to the shadowing effect on the slopes do not disturb the unshadowed scattering coefficient.

We can note that σ_I^S assumes that the correlation between the slopes and the heights is negligible, which explains the deviation between σ_I^S and σ_I^{US}. It will be interesting to compare these in magnitude when the correlation is introduced, because the unshadowed incoherent scattering coefficient σ_I^{US} decreases less slowly than that obtained without correlation σ_{IUN}^{US}.

6. Conclusion

The backscattering from a two-dimensional randomly rough perfectly-conducting surface has been investigated using Kirchhoff's approach (KA) with shadowing effect. For a non-perfectly-conducting surface with a slope and height joint Gaussian stationary process and for any surface height autocorrelation function, the scattering coefficient obtained from the Kirchhoff integral requires eight integrations. This means that the problem cannot be solved analytically and numerically without additional assumptions. Moreover, the introduction of the shadowing function involves that the surface height and slope joint probability density function has to be determined considering the shadowing function.

For a monostatic configuration with a perfectly-conducting surface, seven integrations can be performed analytically for an isotropic surface, instead of six for a two-dimensional surface. With the inclusion of the shadowing function, the number of numerical integrations increases by one.

For an isotropic surface, the simulations show that $k_c = \sqrt{2}kL_c$ and $\sigma_X \tan\theta_s$, where k is the wavenumber, L_c the surface correlation length, σ_X the surface slope standard deviation, are relevant parameters for estimating the agreement between the results obtained from the Kirchhoff approximation and those computed from the stationary phase (SP) and geometrical optics approximations. The Kirchhoff theory is valid if the parameter $kL_c = k_c/\sqrt{2}$ and the surface curvature radius are larger than 2π.

With Gaussian and power surface height autocorrelation functions, the simulations show that the incoherent backscattering coefficient is barely noticeable according to the autocorrelation function. Moreover, it is interesting to note that the stationary phase method underestimated the incoherent backscattering coefficient. This deviation between the results obtained from SP and KA decreases when the surface slope variance decreases.

Assuming that the correlation between the heights and the slopes is negligible (it is often the case for the different spectra we used), it is observed that the backscattering coefficient calculated with shadow is similar to that obtained without shadow. This comes from the fact that when the shadowing effect becomes important, the incoherent component is small. It will be interesting to apply the same method with correlation.

The prospect of this work may be the study of the dielectric surface for one- and two-dimensional surfaces. We have seen that the effect of the shadowing function may be small for a backscattering configuration, but we believe that for a bistatic configuration the effect of the shadowing function cannot be ignored. Since it is very difficult to solve the Kirchhoff integral for any configuration, it will be interesting to use the exposed method for one- and two-dielectric surfaces with the stationary phase approximation in order to verify the previous comment.

Appendix A. Expected values of first order without shadow

This appendix presents the calculus of the following expected values $E_4(\gamma_X)$, $E_4(\gamma'_X)$, $E_4(\gamma_Y)$ and $E_4(\gamma'_Y)$ when the shadowing effect is not investigated. Firstly, the expected value $E_4(\gamma_X)$ is performed, and we show that the others are obtained from $E_4(\gamma_X)$ by variable transformations.

We need to solve the following integral over γ_X:

$$E_4(\gamma_X) = \int_{-\infty}^{\infty} \gamma_X \,\mathrm{d}\gamma_X \left[\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} p(z', z'', \gamma_X, \gamma'_X, \gamma_Y, \gamma'_Y)\,\mathrm{d}\gamma'_X\,\mathrm{d}\gamma_Y\,\mathrm{d}\gamma'_Y\right]. \tag{A1}$$

The term between brackets corresponds to the marginal probability integrated over $\{\gamma'_X, \gamma_Y, \gamma'_Y\}$ and is equal to the probability density $p(z', z'', \gamma_X)$ given by

$$p(z', z'', \gamma_X) = \frac{1}{(2\pi)^{3/2}|[M]|^{1/2}} \exp\left(-\frac{1}{2}[z' z'' \gamma_X][M]^{-1}\begin{bmatrix} z' \\ z'' \\ \gamma_X \end{bmatrix}\right) \tag{A2}$$

where $[M]$ is (12) $[C_{XY}]$ partitioned matrix defined as

$$[M] = \begin{bmatrix} \omega^2 & R_0 & 0 \\ R_0 & \omega^2 & -R_1 \\ 0 & -R_1 & \sigma_X^2 \end{bmatrix}. \tag{A3}$$

The inversion of the matrix $[M]$ leads to

$$p(z', z'', \gamma_X) = \frac{1}{(2\pi)^{3/2}|[M]|^{1/2}} \times \exp\left[-\frac{M_{i11}z'^2 + M_{i22}z''^2 + 2M_{i12}z'z'' + M_{i33}\gamma_X^2 + 2(M_{i13}z' + M_{i23}z'')\gamma_X}{2|[M]|}\right] \tag{A4}$$

with

$$\begin{aligned} &M_{i11} = \omega^2\sigma_X^2(1 - f_1^2) && M_{i12} = -\omega^2\sigma_X^2 f_0 && f_0 = R_0/\omega^2 \\ &M_{i22} = \omega^2\sigma_X^2 && M_{i13} = \omega^3\sigma_X f_0 f_1 && f_1 = -R_1/(\omega\sigma_X) \\ &M_{i33} = \omega^4(1 - f_0^2) && M_{i23} = -\omega^3\sigma_X f_1 && |[M]| = \omega^4\sigma_X^2(1 - f_1^2 - f_0^2) \end{aligned} \tag{A5}$$

where $|[M]|$ is the determinant of $[M]$. Writing that

$$p(z', z'', \gamma_X) = \frac{1}{(2\pi)^{3/2}|[M]|^{1/2}} \exp(-a\gamma_X^2 - 2b\gamma_X - c). \tag{A6}$$

Identifying this equation with (A4) and using the following relationship:

$$\int_{-\infty}^{\infty} \gamma_X \exp(-a\gamma_X^2 - 2b\gamma_X - c)\,\mathrm{d}\gamma_X = -\frac{b\sqrt{\pi}}{a^{3/2}} \exp\left(\frac{b^2}{a} - c\right) \tag{A7}$$

with $a > 0$, we show that the expected value $E_4(\gamma_X)$ is

$$E_4(\gamma_X) = \frac{\sigma_X f_1(z'' - z' f_0)}{\omega(1 - f_0^2)} p(z', z'') \tag{A8}$$

where $p(z', z'')$ is the surface height joint probability expressed as

$$p(z', z'') = \frac{1}{2\pi\omega^2(1 - f_0^2)^{1/2}} \exp\left[-\frac{1}{2\omega^2(1 - f_0^2)}(z'^2 + z''^2 - 2f_0 z' z'')\right]. \tag{A9}$$

The calculation of $b^2/a - c$ in (A7) is not required due to the fact that it has to be equal to the exponential term of $p(z', z'')$. When $f_1 = 0$, which is similar to neglecting the correlation between the heights and the slopes, and since the surface slope mean m_{γ_X} is assumed to be equal to zero, we have $E_4(\gamma_X) = m_{\gamma_X} = 0$.

The computation of $E_4(\gamma'_X)$ is obtained from the marginal probability $p(z', z'', \gamma'_X)$ characterized by the following covariance matrix:

$$[M'] = \begin{bmatrix} \omega^2 & R_0 & R_1 \\ R_0 & \omega^2 & 0 \\ R_1 & 0 & \sigma_X^2 \end{bmatrix}. \tag{A10}$$

Comparing (A10) with (A3), the covariance matrix $[M']$ is obtained from $[M]$ by swapping z' in z'' and R_1 in $-R_1$. Therefore, from (A8) we obtain

$$E_4(\gamma'_X) = -\frac{\sigma_X f_1(z' - z'' f_0)}{\omega(1 - f_0^2)} p(z', z''). \tag{A11}$$

Consequently,

$$E_4(\gamma_X) + E_4(\gamma'_X) = \frac{\sigma_X f_1(z'' - z')}{\omega(1 - f_0)} p(z', z''). \tag{A12}$$

Using the same method as the derivations of $\{E_4(\gamma_X), E_4(\gamma'_X)\}$, the expected values $\{E_4(\gamma_Y), E_4(\gamma'_Y)\}$ are computed from (A12) by swapping f_1 in $f_{16} = -C_{16}/(\omega\sigma_Y)$, thus

$$E_4(\gamma_Y) + E_4(\gamma'_Y) = \frac{\sigma_Y f_{16}(z'' - z')}{\omega(1 - f_0)} p(z', z''). \tag{A13}$$

Appendix B. Expected values of second order without shadow

This appendix presents the calculus of the following expected values $E_4(\gamma_X\gamma'_Y)$, $E_4(\gamma_Y\gamma'_X)$, $E_4(\gamma_X\gamma'_X)$ and $E_4(\gamma_Y\gamma'_Y)$ without a shadowing effect. Firstly, the expected value $E_4(\gamma_X\gamma'_Y)$ is performed, and we show that the others are obtained from $E_4(\gamma_X\gamma'_Y)$ by variable transformations.

We need to solve the following integral over $\gamma_X\gamma'_Y$:

$$E_4(\gamma_X\gamma'_Y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \gamma_X\gamma'_Y \, \mathrm{d}\gamma_X \, \mathrm{d}\gamma'_Y \left[\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} p(z', z'', \gamma_X, \gamma'_X, \gamma_Y, \gamma'_Y) \, \mathrm{d}\gamma'_X \, \mathrm{d}\gamma_Y\right]. \tag{B1}$$

The term between brackets corresponds to the marginal probability integrated over $\{\gamma'_X, \gamma_Y\}$ and equal to the probability density $p(z', z'', \gamma_X, \gamma'_Y)$ given by

$$p(z', z'', \gamma_X, \gamma'_Y) = \frac{1}{(2\pi)^2|[C]|^{1/2}} \exp\left(-\tfrac{1}{2}[z'z''\gamma_X\gamma'_Y][C]^{-1}\begin{bmatrix} z' \\ z'' \\ \gamma_X \\ \gamma'_Y \end{bmatrix}\right) \tag{B2}$$

where $[C]$ is (12) $[C_{XY}]$ partitioned matrix defined as

$$[C] = \begin{bmatrix} \omega^2 & R_0 & 0 & C_{16} \\ R_0 & \omega^2 & -R_1 & 0 \\ 0 & -R_1 & \sigma_X^2 & -C_{36} \\ C_{16} & 0 & -C_{36} & \sigma_Y^2 \end{bmatrix}. \tag{B3}$$

The inversion of the covariance matrix $[C]$ leads to

$$p(z', z'', \gamma_X, \gamma_Y') = \frac{1}{(2\pi)^2|[C]|^{1/2}} \exp\left[-\frac{z'^2C_{i11} + z''^2C_{i22} + \gamma_X^2C_{i33} + \gamma_Y'^2C_{i44}}{2|[C]|} - \frac{2z'z''C_{i12} + 2\gamma_X\gamma_Y'C_{i34} + 2z'(\gamma_XC_{i13} + \gamma_Y'C_{i14}) + 2z''(\gamma_XC_{i23} + \gamma_Y'C_{i24})}{2|[C]|}\right] \quad \text{(B4)}$$

with

$$[C]^{-1} = \frac{1}{|[C]|}\begin{bmatrix} C_{i11} & C_{i12} & C_{i13} & C_{i14} \\ C_{i12} & C_{i22} & C_{i23} & C_{i24} \\ C_{i13} & C_{i23} & C_{i33} & C_{i34} \\ C_{i14} & C_{i24} & C_{i34} & C_{i44} \end{bmatrix} \quad \text{(B5)}$$

where $\{C_{ijk}, |[C]|\}$ are given by

$$\begin{aligned} C_{i11} &= \omega^2(\sigma_X^2\sigma_Y^2 - C_{36}^2) - \sigma_Y^2R_1^2 & C_{i13} &= -\sigma_Y^2R_0R_1 - \omega^2C_{16}C_{36} \\ C_{i22} &= \omega^2(\sigma_X^2\sigma_Y^2 - C_{36}^2) - \sigma_X^2C_{16}^2 & C_{i24} &= \sigma_X^2R_0C_{16} + \omega^2R_1C_{36} \\ C_{i12} &= R_0(C_{36}^2 - \sigma_X^2\sigma_Y^2) - R_1C_{16}C_{36} & C_{i14} &= C_{16}(R_1^2 - \omega^2\sigma_X^2) - R_0R_1C_{36} \\ C_{i33} &= \sigma_Y^2(\omega^4 - R_0^2) - \omega^2C_{16}^2 & C_{i23} &= R_1(\omega^2\sigma_Y^2 - C_{16}^2) + R_0C_{16}C_{36} \\ C_{i44} &= \sigma_X^2(\omega^4 - R_0^2) - \omega^2R_1^2 & |[C]| &= \frac{C_{i33}C_{i34} - C_{i34}^2}{\omega^4 - R_0^2} \\ C_{i34} &= C_{36}(\omega^4 - R_0^2) + R_0R_1C_{16}. \end{aligned} \quad \text{(B6)}$$

Using (A7) and the same form as (A6) according to γ_Y', the integration over γ_Y' leads to

$$E_4(\gamma_X\gamma_Y') = \frac{-\sqrt{\pi}}{(2\pi)^2|[C]|^{1/2}a^{3/2}} \int_{-\infty}^{\infty} \gamma_X(b_1 + \gamma_Xb_2) \exp\left(\frac{b^2}{a} - c\right) \mathrm{d}\gamma_X \quad \text{(B7)}$$

with

$$\begin{aligned} a &= \frac{C_{i44}}{2|[C]|} \\ b_1 &= \frac{z'C_{i14} + z''C_{i24}}{2|[C]|} \qquad b_2 = \frac{C_{i34}}{2|[C]|} \qquad b = b_1 + \gamma_Xb_2 \\ c &= \frac{z'^2C_{i11} + z''^2C_{i22} + 2z'z''C_{i12} + 2\gamma_X(z'C_{i13} + z''C_{i23}) + \gamma_X^2C_{i33}}{2|[C]|}. \end{aligned} \quad \text{(B8)}$$

To perform the integration over γ_X, the exponential term $b^2/a - c$ is written as $-a'\gamma_X^2 - 2b'\gamma_X - c'$, and using (A7) and the following expression:

$$\int_{-\infty}^{\infty} \gamma_X^2 \exp(-a'\gamma_X^2 - 2b'\gamma_X - c')\,\mathrm{d}\gamma_X = \frac{\sqrt{\pi}}{2a'^{5/2}} \exp\left(\frac{b'^2}{a'} - c'\right)(a' + 2b'^2) \quad \text{(B9)}$$

with

$$\begin{aligned} a' &= \frac{C_{i33}C_{i44} - C_{i34}^2}{2|[C]|C_{i44}} \\ b' &= \frac{z'(C_{i44}C_{i13} - C_{i34}C_{i14}) + z''(C_{i44}C_{i23} - C_{i34}C_{i24})}{2|[C]|C_{i44}} \end{aligned} \quad \text{(B10)}$$

we show that

$$E_4(\gamma_X\gamma_Y') = \frac{b_2}{8\pi|[C]|^{1/2}(aa')^{3/2}}\left[2b\left(\frac{b_1}{b_2}-\frac{b'}{a'}\right)-1\right]\exp\left(\frac{b'^2}{a'}-c'\right). \quad \text{(B11)}$$

The determination of $b'^2/a' - c'$ is not required because it corresponds to the exponential term of the surface height joint probability density $p(z', z'')$ given by (A9). This involves that $\exp(b'^2/a' - c') = 2\pi\sqrt{\omega^4 - R_0^2}\, p(z', z'')$ with $R_0 = \omega^2 f_0$. Substituting (B8) and (B10) into (B11), we have

$$\begin{aligned}E_4(\gamma_X\gamma_Y') = \sqrt{\omega^4 - R_0^2}\left(\frac{|[C]|}{C_{i33}C_{i34} - C_{i34}^2}\right)^{3/2} p(z', z'')\big\{-C_{i34} + \big[[z'(C_{i34}C_{i14} - C_{i44}C_{i13}) \\ +z''(C_{i34}C_{i24} - C_{i44}C_{i23})][z'(C_{i34}C_{i13} - C_{i33}C_{i14}) \\ +z''(C_{i34}C_{i23} - C_{i33}C_{i24})]\big]\big[|[C]|(C_{i33}C_{i44} - C_{i34}^2)\big]^{-1}\big\}. \quad \text{(B12)}\end{aligned}$$

Substituting (B6) into (B12), we show

$$E_4(\gamma_X\gamma_Y') = \frac{p(z', z'')}{\omega^4 - R_0^2}\left\{\frac{C_{16}R_1(z'R_0 - z''\omega^2)(z'\omega^2 - z''R_0)}{\omega^4 - R_0^2} - [C_{16}R_1R_0 + C_{36}(\omega^4 - R_0^2)]\right\}. \quad \text{(B13)}$$

If $R_1 = 0$ and $C_{16} = 0$, then $E_4(\gamma_X\gamma_Y') = -C_{36}p(z', z'')$, which corresponds to the slope cross-correlation when the correlation between the heights and the slopes is neglected.

The computation of $E_4(\gamma_X\gamma_Y')$ is obtained from the marginal probability $p(z', z'', \gamma_X', \gamma_Y)$ characterized by the following covariance matrix:

$$[C'] = \begin{bmatrix} \omega^2 & R_0 & R_1 & 0 \\ R_0 & \omega^2 & 0 & -C_{16} \\ R_1 & 0 & \sigma_X^2 & -C_{36} \\ 0 & -C_{16} & -C_{36} & \sigma_Y^2 \end{bmatrix}. \quad \text{(B14)}$$

Comparing (B14) with (B3), the covariance matrix $[C']$ is obtained from $[C]$ by swapping z' in z'' and $\{R_1, C_{16}\}$ in $\{-R_1, -C_{16}\}$. Therefore, from (B13) we obtain

$$E_4(\gamma_Y\gamma_X') = E_4(\gamma_X\gamma_Y'). \quad \text{(B15)}$$

The determination of $E_4(\gamma_X\gamma_X')$ is obtained from the marginal probability $p(z', z'', \gamma_X, \gamma_X')$ characterized by the following covariance matrix:

$$[C_1] = \begin{bmatrix} \omega^2 & R_0 & 0 & R_1 \\ R_0 & \omega^2 & -R_1 & 0 \\ 0 & R_1 & \sigma_X^2 & -R_2 \\ R_1 & 0 & -R_2 & \sigma_X^2 \end{bmatrix}. \quad \text{(B16)}$$

Comparing (B16) with (B3), the covariance matrix $[C_1]$ is obtained from $[C]$ by making $\{C_{16} = R_1, C_{36} = R_2, \sigma_Y^2 = \sigma_X^2\}$. Therefore, from (B13) we have

$$E_4(\gamma_X\gamma_X') = \frac{p(z', z'')}{\omega^4 - R_0^2}\left\{\frac{R_1^2(z'R_0 - z''\omega^2)(z'\omega^2 - z''R_0)}{\omega^4 - R_0^2} - [R_1^2R_0 + R_2(\omega^4 - R_0^2)]\right\}. \quad \text{(B17)}$$

The expected value $E_4(\gamma_Y\gamma'_Y)$ is computed from the covariance matrix

$$[C'_1] = \begin{bmatrix} \omega^2 & R_0 & 0 & C_{16} \\ R_0 & \omega^2 & -C_{16} & 0 \\ 0 & -C_{16} & \sigma_Y^2 & -C_{56} \\ C_{16} & 0 & -C_{56} & \sigma_Y^2 \end{bmatrix}. \tag{B18}$$

Comparing (B18) with (B16), the covariance matrix $[C'_1]$ is obtained from $[C_1]$ by making $\{R_1 = C_{16}, R_2 = C_{56}\}$. Therefore, from (B17) we have

$$E_4(\gamma_Y\gamma'_Y) = \frac{p(z', z'')}{\omega^4 - R_0^2}\left\{\frac{C_{16}^2(z'R_0 - z''\omega^2)(z'\omega^2 - z''R_0)}{\omega^4 - R_0^2} - [C_{16}^2 R_0 + C_{56}(\omega^4 - R_0^2)]\right\}. \tag{B19}$$

Appendix C. Characteristic function with shadowing effect

This appendix presents the derivation of the surface height joint characteristic function, with a shadowing effect, determined from the method exposed in subsection 4.2 by making $\vec{V}_{DXY}^T = [\xi'\xi'']$, $\vec{V}_{XY}^T = [z'z'']$, and $[C_{XY}] = [H]$.

The $[V]$ unitary eigenvectors and the $[\Sigma]$ eigenvalues matrix of (31) $[H]$ is then

$$[\Sigma] = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \qquad [V] = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \qquad \begin{cases} \lambda_1 = \omega^2(1 - f_0) \\ \lambda_2 = \omega^2(1 + f_0). \end{cases} \tag{C1}$$

Since the covariance matrix is Hermitian the eigenvalues $\lambda_{i\in[1;2]} \geqslant 0$. Using (29) with $n = -\frac{1}{2}$ and inverting (27), the height uncorrelated samples $\{\xi', \xi''\}$ are expressed from the height correlated samples $\{z', z''\}$ as

$$\begin{bmatrix} \xi' \\ \xi'' \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{\lambda_1}} + \frac{1}{\sqrt{\lambda_2}} & \frac{1}{\sqrt{\lambda_1}} - \frac{1}{\sqrt{\lambda_2}} \\ \frac{1}{\sqrt{\lambda_1}} - \frac{1}{\sqrt{\lambda_2}} & \frac{1}{\sqrt{\lambda_1}} + \frac{1}{\sqrt{\lambda_2}} \end{bmatrix}\begin{bmatrix} z' \\ z'' \end{bmatrix}. \tag{C2}$$

Substituting (C2) into (26) and using (30), we show that the surface height joint probability density with shadowing effect is expressed as follows:

$$p_S(z', z'') = p(z', z'')\left\{\left[1 - \tfrac{1}{2}\operatorname{erfc}\left(\frac{b_1 z' - b_2 z''}{\omega\sqrt{2}}\right)\right]\left[1 - \tfrac{1}{2}\operatorname{erfc}\left(\frac{b_1 z'' - b_2 z'}{\omega\sqrt{2}}\right)\right]\right\}^{\Lambda} \tag{C3}$$

with

$$\begin{aligned} b_1 &= \frac{1}{2}\left[\frac{1}{(1 - f_0)^{1/2}} + \frac{1}{(1 + f_0)^{1/2}}\right] \\ b_2 &= \frac{1}{2}\left[\frac{1}{(1 - f_0)^{1/2}} - \frac{1}{(1 + f_0)^{1/2}}\right]. \end{aligned} \tag{C4}$$

The second term of (C3) denotes the shadowing effect. If $f_0 = 0$ corresponding to the heights uncorrelated case, then $\{b_1 = 1, b_2 = 0\}$, the second term of (C3) then becomes equal to that of (26), with $\omega = 1$ and $\{\xi' = z', \xi'' = z''\}$.

Substituting (C3) into (21), the surface height joint characteristic function with shadow becomes

$$\chi_{S1} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} p(z', z'')\left\{\left[1 - \frac{1}{2}\,\mathrm{erfc}\left(\frac{b_1 z' - b_2 z''}{\omega\sqrt{2}}\right)\right]\left[1 - \frac{1}{2}\,\mathrm{erfc}\left(\frac{b_1 z'' - b_2 z'}{\omega\sqrt{2}}\right)\right]\right\}^{\Lambda} \times \exp[\mathrm{i}q_z(z' - z'')]\,\mathrm{d}z'\,\mathrm{d}z''. \tag{C5}$$

In order to simplify the first double integral χ_{S1}, the following variable transformations are used:

$$Z' = \frac{b_1 z' - b_2 z''}{\omega\sqrt{2}} \qquad Z'' = \frac{b_1 z'' - b_2 z'}{\omega\sqrt{2}} \tag{C6}$$

which leads to

$$z' = \omega\sqrt{2}\frac{b_1 Z' + b_2 Z''}{b_1^2 - b_2^2} \qquad z'' = \omega\sqrt{2}\frac{b_2 Z' + b_1 Z''}{b_1^2 - b_2^2}. \tag{C7}$$

Substituting (C6) and (C7) into χ_{S1} with the Jacobian equal to $2\omega^2(1 - f_0^2)^{1/2}$ and using the definitions of (C4) $\{b_1, b_2\}$, we prove that

$$\chi_{S1} = \left|F(q_z\omega\sqrt{2}\sqrt{1 - f_0})\right|^2 \tag{C8}$$

with

$$F(\cdots) = \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} \exp[(\cdots)Z']\exp(-Z'^2)\left[1 - \tfrac{1}{2}\,\mathrm{erfc}(Z')\right]^{\Lambda}\,\mathrm{d}Z'. \tag{C9}$$

The variable transformations allow transformation of a double integral into two independent simple conjugate integrals. If the shadowing function is ignored which is similar to having $\Lambda = 0$, then the integration over Z' can be determined analytically and gives $\exp[-q_z^2\omega^2(1 - f_0)]$ corresponding to the χ_1 term of (23).

Appendix D. Expected values of first and second orders with shadowing effect

This appendix presents the calculus of the expected values $E_{S4}(\cdots)$ with a shadowing effect defined by (33).

The expected value $E_{S4}(1)$ is defined as follows:

$$E_{S4}(1) = \int_{-\infty}^{\mu}\int_{-\infty}^{\mu} \mathrm{d}\gamma_X\,\mathrm{d}\gamma'_X\left[\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} p(\gamma_X, \gamma'_X, \gamma_Y, \gamma'_Y)\,\mathrm{d}\gamma_Y\,\mathrm{d}\gamma'_Y\right]. \tag{D1}$$

The term between brackets corresponds to the marginal probability integrated over $\{\gamma_Y, \gamma'_Y\}$ and is equal to the probability density $p(\gamma_X, \gamma'_X)$ expressed as

$$p(\gamma_X, \gamma'_X) = \frac{1}{2\pi\sigma_X^2(1 - f_2^2)^{1/2}}\exp\left[-\frac{1}{2\sigma_X^2(1 - f_2^2)}(\gamma_X^2 + \gamma_X'^2 - 2f_2\gamma_X\gamma'_X)\right]. \tag{D2}$$

Substituting (D2) into (D1) and applying the following variable transformations:

$$\begin{aligned}\gamma_X &= \sigma_X[X(1 + f_2)^{1/2} - X'(1 - f_2)^{1/2}]\\ \gamma'_X &= \sigma_X[X(1 + f_2)^{1/2} + X'(1 - f_2)^{1/2}]\end{aligned} \tag{D3}$$

the integral (D1) with the Jacobian equal to $2\sigma_X^2(1 - f_2^2)^{1/2}$ becomes

$$E_{S4}(1) = \frac{1}{\pi}\int_{-\infty}^{\mu/\sigma_X(1+f_2)^{1/2}} \mathrm{d}X \int_{-\infty}^{\infty} \exp(-X^2 - X'^2)\,\mathrm{d}X'. \tag{D4}$$

Performing the integration over $\{X, X'\}$, we obtain

$$E_{S4}(1) = \tfrac{1}{2}[1 + \mathrm{erf}(\mu_0)] \qquad \mu_0 = \frac{\mu}{\sigma_X(1+f_2)^{1/2}}. \tag{D5}$$

Using the same method as previously, we prove that the expected value $E_{S4}(\gamma'_X\gamma_X)$ is

$$E_{S4}(\gamma'_X\gamma_X) = \frac{\sigma_X^2 f_2}{2}[1 + \mathrm{erf}(\mu_0)] - \frac{\sigma_X^2(1+f_2)\mu_0\exp(-\mu_0^2)}{2\sqrt{\pi}}. \tag{D6}$$

If the shadowing function is ignored, which is the same as making $\mu_0 \to \infty$, then $E_{S4}(\gamma'_X\gamma_X) = \sigma_X^2 f_2$.

The expected value $E_{S4}(\gamma'_Y)$ is defined as

$$E_{S4}(\gamma'_Y) = \int_{-\infty}^{\mu} \mathrm{d}\gamma_X \int_{-\infty}^{\mu} \mathrm{d}\gamma'_X \int_{-\infty}^{\infty} \gamma'_Y\,\mathrm{d}\gamma'_Y \left[\int_{-\infty}^{\infty} p(\gamma_X, \gamma'_X, \gamma_Y, \gamma'_Y)\,\mathrm{d}\gamma_Y\right]. \tag{D7}$$

The term between brackets corresponds to $p(\gamma_X, \gamma'_X, \gamma'_Y)$, the marginal probability integrated over γ_Y characterized by (31) $[S]$ partitioned matrix is expressed as

$$[S'_1] = \begin{bmatrix} \sigma_X^2 & -R_2 & -C_{36} \\ -R_2 & \sigma_X^2 & \sigma_{XY}^2 \\ -C_{36} & \sigma_{XY}^2 & \sigma_Y^2 \end{bmatrix}. \tag{D8}$$

Applying the same method as appendix A, and performing the integration over γ'_Y we show that

$$E_{S4}(\gamma'_Y) = \frac{1}{\sigma_X^2(1-f_2^2)} \int_{-\infty}^{\mu}\int_{-\infty}^{\mu} [\gamma'_X(\sigma_{XY}^2 - \sigma_X\sigma_Y f_2 f_{36}) + \gamma_X(\sigma_X\sigma_Y f_{36} - \sigma_{XY}^2 f_2)]\, p(\gamma_X, \gamma'_X)\,\mathrm{d}\gamma_X\,\mathrm{d}\gamma'_X \tag{D9}$$

where $\{f_2, f_{36}\}$ are given by (19*a*).

The expected value $E_{S4}(\gamma_X\gamma'_Y)$ is given by

$$E_{S4}(\gamma_X\gamma'_Y) = \int_{-\infty}^{\mu}\int_{-\infty}^{\mu}\int_{-\infty}^{\infty} \gamma_X\gamma'_Y\, p(\gamma_X, \gamma'_X, \gamma'_Y)\,\mathrm{d}\gamma_X\,\mathrm{d}\gamma'_X\,\mathrm{d}\gamma'_Y. \tag{D10}$$

From (D9), the integration over γ'_Y leads to

$$E_{S4}(\gamma_X\gamma'_Y) = \frac{1}{\sigma_X^2(1-f_2^2)} \int_{-\infty}^{\mu}\int_{-\infty}^{\mu} \gamma_X[\gamma'_X(\sigma_{XY}^2 - \sigma_X\sigma_Y f_2 f_{36}) + \gamma_X(\sigma_X\sigma_Y f_{36} - \sigma_{XY}^2 f_2)] \times p(\gamma_X, \gamma'_X)\,\mathrm{d}\gamma_X\,\mathrm{d}\gamma'_X. \tag{D11}$$

Using the variable transformation (D3), and performing the integration over $\{X, X'\}$, we have

$$E_{S4}(\gamma_X\gamma'_Y) = \frac{\sigma_X\sigma_Y f_{36}}{2}[1 + \mathrm{erf}(\mu_0)] - (\sigma_{XY}^2 + \sigma_X\sigma_Y f_{36})\frac{\mu_0\exp(-\mu^2)}{2\sqrt{\pi}}. \tag{D12}$$

If the shadowing function is ignored, which is the same as making $\mu_0 \to \infty$, then $E_{S4}(\gamma_X\gamma'_Y) = \sigma_X\sigma_Y f_{36} = -C_{36}$, corresponding to the cross-slope correlation.

The last expected value to calculate is $E_{S4}(\gamma_Y\gamma'_Y)$ defined as

$$E_{S4}(\gamma_Y\gamma'_Y) = \int_{-\infty}^{\mu} \mathrm{d}\gamma_X \int_{-\infty}^{\mu} \mathrm{d}\gamma'_X \left[\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \gamma_Y\gamma'_Y\, p(\gamma_X, \gamma'_X, \gamma_Y, \gamma'_Y)\,\mathrm{d}\gamma_Y\,\mathrm{d}\gamma'_Y\right]. \tag{D13}$$

The integrations over $\{\gamma_Y, \gamma'_Y\}$ are similar to those made in (B1), with $\{z' = \gamma_X, z'' = \gamma'_X, \gamma_X = \gamma_Y\}$. However, in this case, the covariance matrix $[S]$ is defined from (31) instead of (B3). Consequently, from (B12), the integrations over $\{\gamma_Y, \gamma'_Y\}$ lead to

$$\sqrt{\sigma_X^4 - R_2^2}\left(\frac{|[S]|}{S_{i33}S_{i34} - S_{i34}^2}\right)^{3/2} p(\gamma_X, \gamma'_X)$$
$$\times\big\{-S_{i34} + \big[[\gamma_X(S_{i34}S_{i14} - S_{i44}S_{i13}) + \gamma'_X(S_{i34}S_{i24} - S_{i44}S_{i23})]$$
$$\times[\gamma'_X(S_{i34}S_{i13} - S_{i33}S_{i14}) + \gamma_X(S_{i34}S_{i23} - S_{i33}S_{i24})]\big]$$
$$\times\big[|[S]|(S_{i33}S_{i44} - S_{i34}^2)\big]^{-1}\big\} \tag{D14}$$

where S_{ijk} are the elements of the $[S]$ inverse covariance matrix given by

$$\begin{aligned}
S_{i11} &= S_{i22} = \sigma_X^2(\sigma_Y^4 - C_{56}^2) - \sigma_Y^2(\sigma_{XY}^4 + C_{36}^2) + 2\sigma_{XY}^2 C_{36}C_{56}\\
S_{i33} &= S_{i44} = \sigma_Y^2(\sigma_X^4 - R_2^2) + \sigma_X^2(\sigma_{XY}^4 + C_{36}^2) + 2\sigma_{XY}^2 R_2 C_{36}\\
S_{i12} &= R_2(\sigma_Y^4 - C_{56}^2) + C_{56}(\sigma_{XY}^4 + C_{36}^2) - 2\sigma_{XY}^2\sigma_X^2 C_{36}\\
S_{i34} &= C_{56}(\sigma_X^4 - R_2^2) + R_2(\sigma_{XY}^4 + C_{36}^2) - 2\sigma_{XY}^2\sigma_X^2 C_{36}\\
S_{i13} &= S_{i24} = R_2(\sigma_Y^2 C_{36} - \sigma_{XY}^2 C_{56}) + \sigma_X^2(C_{36}C_{56} - \sigma_Y^2\sigma_{XY}^2) + \sigma_{XY}^2(\sigma_{XY}^4 - C_{36}^2)\\
S_{i14} &= S_{i23} = R_2(C_{36}C_{56} - \sigma_Y^2\sigma_{XY}^2) + \sigma_X^2(\sigma_Y^2 C_{36} - \sigma_{XY}^2 C_{56}) + C_{36}(\sigma_{XY}^4 - C_{36}^2)\\
|[S]| &= (S_{i33}^2 - S_{i34}^2)/(\sigma_X^4 - R_2^2).
\end{aligned} \tag{D15}$$

Substituting (D15) into (D14), we show that

$$E_{S4}(\gamma_Y\gamma'_Y) = \frac{1}{\sigma_X^4 - R_2^2}\int_{-\infty}^{\mu}\int_{-\infty}^{\mu} p(\gamma_X, \gamma'_X)$$
$$\times\big\{-[C_{56}(\sigma_X^4 - R_2^2) + R_2(\sigma_{XY}^4 + C_{36}^2) - 2\sigma_{XY}^2\sigma_X^2 C_{36}]$$
$$+\big[[(C_{36}R_2 - \sigma_{XY}^2\sigma_X^2)\gamma_X + (\sigma_X^2 C_{36} - \sigma_{XY}^2 R_2)\gamma'_X][(C_{36}R_2 - \sigma_{XY}^2\sigma_X^2)\gamma'_X$$
$$+(\sigma_X^2 C_{36} - \sigma_{XY}^2 R_2)\gamma_X]\big]\big[\sigma_X^4 - R_2^2\big]^{-1}\big\}\,\mathrm{d}\gamma_X\,\mathrm{d}\gamma'_X. \tag{D16}$$

Using the variable transformation (D3), and performing the integration over $\{X, X'\}$, we obtain

$$E_{S4}(\gamma_Y\gamma'_Y) = -\frac{C_{56}}{2}[1 + \mathrm{erf}(\mu_0)] - \frac{(\sigma_{XY}^2 - C_{36})^2}{\sigma_X^2 - R_2}\frac{\mu_0\exp(-\mu_0^2)}{2\sqrt{\pi}}. \tag{D17}$$

The use of (19*a*) leads to

$$E_{S4}(\gamma_Y\gamma'_Y) = \frac{\sigma_Y^2 f_{56}}{2}[1 + \mathrm{erf}(\mu_0)] - \frac{(\sigma_{XY}^2 + \sigma_X\sigma_Y f_{36})^2}{\sigma_X^2(1 + f_2)}\frac{\mu_0\exp(-\mu_0^2)}{2\sqrt{\pi}}. \tag{D18}$$

If the shadowing function is ignored, which is the same as making $\mu_0 \to \infty$, then $E_{S4}(\gamma_Y\gamma'_Y) = -C_{56}$.

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